RRV-30

U.S. Department
of Transportation
**Federal Railroad
Administration**

Procedure to Establish Inspection Intervals of Regularly Maintained In-Service Units Subject to Failure

Office of Research
and Development
Washington, DC 20590

Research and Special Programs Administration
Volpe National Transportation Systems Center
Cambridge, MA 02142-1093

DOT/FRA/ORD-96/04
DOT-VNTSC-FRA-95-9

Final Report
May 1996

This document is available to the public through the National Technical Information Service, Springfield, VA 22161

12 - Safety

REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1. AGENCY USE ONLY (Leave blank)

2. REPORT DATE
May 1996

3. REPORT TYPE AND DATES COVERED
Final Report
June 1990 - June 1995

4. TITLE AND SUBTITLE
Procedure to Establish Inspection Intervals of Regularly Maintained In-Service Units Subject to Failure

5. FUNDING NUMBERS
RR528/R5025

6. AUTHOR(S)
Peter H. Mengert, Joseph Davin, Herbert Weinstock

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
U.S. Department of Transportation
Research & Special Programs Administration
Volpe National Transportation Systems Center
Cambridge, MA 02142-1093

8. PERFORMING ORGANIZATION REPORT NUMBER
DOT-VNTSC-FRA-95-9

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
U.S. Department of Transportation
Federal Railroad Administration
Office of Research and Development
Washington, DC 20590

10. SPONSORING/MONITORING AGENCY REPORT NUMBER
DOT/FRA/ORD-96/04

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT

This document is available to the public through the National Technical Information Service, Springfield, VA 22161

12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)

Procedures have been developed for determining the period between regular inspections that is required to ensure, with a specified level of confidence, that no more than a certain percentage of the units of a population that is in service is failed. The procedure assumes that the units have a uniform random failure rate and that the population is divided into many equal sized groups. These groups are inspected sequentially at a uniform rate over the inspection interval, repaired if necessary, and returned to service. A new inspection interval is calculated based on the number found failed in an initial inspection interval. Comparisons of repeated simulations of the percentage of in-service units of the population using randomly generated data have determined that the procedure is useful when the population is greater than 100 units and the percentage of the population found failed at inspection is less than 1%. These procedures, combined with engineering data and experience, may be applied to the definition of approaches for ensuring safety of equipment in railroad operations.

14. SUBJECT TERMS
Inspection, Reliability, Failure Rate, Repair, Probability, Simulation, Exponential Failure, Inspection Interval, Statistical Formula, Level of Confidence

15. NUMBER OF PAGES
84

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified	Unclassified	

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. 239-18
298-102

PREFACE

This document describes procedures for determining the period between regular inspections in order to ensure that no more than a certain percentage of units of a selected population in service fail. These procedures assume that the units have a uniform random failure rate and that the population is divided into many groups. These groups are inspected sequentially at a uniform rate over the inspection interval, repaired if necessary, and returned to service. These procedures, combined with engineering data and experience, may be applied to the definition of approaches for ensuring the safety of equipment in railroad operations. These procedures were developed under the Federal Railroad Adminstration (FRA) Rail Equipment Safety Program (RR428) being conducted at the Volpe Center.

METRIC/ENGLISH CONVERSION FACTORS

ENGLISH TO METRIC

LENGTH (APPROXIMATE)

1 inch (in)	= 2.5 centimeters (cm)
1 foot (ft)	= 30 centimeters (cm)
1 yard (yd)	= 0.9 meter (m)
1 mile (mi)	= 1.6 kilometers (km)

AREA (APPROXIMATE)

1 square inch (sq in, in^2)	= 6.5 square centimeters (cm^2)
1 square foot (sq ft, ft^2)	= 0.09 square meter (m^2)
1 square yard (sq yd, yd^2)	= 0.8 square meter (m^2)
1 square mile (sq mi, mi^2)	= 2.6 square kilometers (km^2)
1 acre = 0.4 hectare (he)	= 4,000 square meters (m^2)

MASS - WEIGHT (APPROXIMATE)

1 ounce (oz)	= 28 grams (gm)
1 pound (lb)	= 0.45 kilogram (kg)
1 short ton = 2,000 pounds (lb)	= 0.9 tonne (t)

VOLUME (APPROXIMATE)

1 teaspoon (tsp)	= 5 milliliters (ml)
1 tablespoon (tbsp)	= 15 milliliters (ml)
1 fluid ounce (fl oz)	= 30 milliliters (ml)
1 cup (c)	= 0.24 liter (l)
1 pint (pt)	= 0.47 liter (l)
1 quart (qt)	= 0.96 liter (l)
1 gallon (gal)	= 3.8 liters (l)
1 cubic foot (cu ft, ft^3)	= 0.03 cubic meter (m^3)
1 cubic yard (cu yd, yd^3)	= 0.76 cubic meter (m^3)

TEMPERATURE (EXACT)

[(x-32)(5/9)] °F = y °C

METRIC TO ENGLISH

LENGTH (APPROXIMATE)

1 millimeter (mm)	= 0.04 inch (in)
1 centimeter (cm)	= 0.4 inch (in)
1 meter (m)	= 3.3 feet (ft)
1 meter (m)	= 1.1 yards (yd)
1 kilometer (k)	= 0.6 mile (mi)

AREA (APPROXIMATE)

1 square centimeter (cm^2)	= 0.16 square inch (sq in, in^2)
1 square meter (m^2)	= 1.2 square yards (sq yd, yd^2)
1 square kilometer (km^2)	= 0.4 square mile (sq mi, mi^2)
10,000 square meters (m^2)	= 1 hectare (he) = 2.5 acres

MASS - WEIGHT (APPROXIMATE)

1 gram (gm)	= 0.036 ounce (oz)
1 kilogram (kg)	= 2.2 pounds (lb)
1 tonne (t) = 1,000 kilograms (kg)	= 1.1 short tons

VOLUME (APPROXIMATE)

1 milliliter (ml)	= 0.03 fluid ounce (fl oz)
1 liter (l)	= 2.1 pints (pt)
1 liter (l)	= 1.06 quarts (qt)
1 liter (l)	= 0.26 gallon (gal)
1 cubic meter (m^3)	= 36 cubic feet (cu ft, ft^3)
1 cubic meter (m^3)	= 1.3 cubic yards (cu yd, yd^3)

TEMPERATURE (EXACT)

[(9/5) y + 32] °C = x °F

QUICK INCH - CENTIMETER LENGTH CONVERSION

Inches	0	1	2	3	4	5

Centimeters	0	1	2	3	4	5	6	7	8	9	10	11	12	13

QUICK FAHRENHEIT - CELSIUS TEMPERATURE CONVERSION

°F	-40°	-22°	-4°	14°	32°	50°	68°	86°	104°	122°	140°	158°	176°	194°	212°
°C	-40°	-30°	-20°	-10°	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°

For more exact and or other conversion factors, see NBS Miscellaneous Publication 286, Units of Weights and Measures. Price $2.50 SD Catalog No. C13 10286

Updated 1/23/95

TABLE OF CONTENTS

Section **Page**

1. BACKGROUND 1
2. OBJECTIVE 1
3. PROBABILITY OF FAILURE OF A SINGLE UNIT 2
4. INSPECTION SCHEDULE AND INSPECTION INTERVAL PHASE-IN 3
5. EXPECTED VALUE OF FAILED UNITS AND THE UPPER PREDICTION LIMIT 5
6. SIMULATION TESTING 8
7. TYPICAL RESULTS OF THE ANALYSIS 9
8. SOLUTION RESULTS APPLIED TO A HYPOTHETICAL MAINTENANCE PROBLEM 22
9. LIMITATIONS 28

APPENDIX A. DERIVATION OF THE FORMULA FOR F_U A-1
APPENDIX B. DERIVATION OF VARIANCE FORMULAS B-1
APPENDIX C. DETERMINATION OF F_U AND T_2 . . . . C-1
APPENDIX D. SIMULATION-TESTING OF THE FORMULA FOR F_U D-1
APPENDIX E. RESULTS OF SEVERAL SIMULATION TEST RUNS E-1
APPENDIX F. UPDATE OF THE INSPECTION INTERVAL F-1
APPENDIX G. COMMENTS G-1

LIST OF ILLUSTRATIONS

Figure		Page
4-1.	STAGES IN AN INSPECTION SCHEDULE	4
7-1A.	RATIO OF NEW INSPECTION INTERVAL TO EARLIER INSPECTION INTERVAL VS. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION. THE RATIO ENSURES THAT WITH A 95% CONFIDENCE NO MORE THAN 5% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME	10
7-1B.	RATIO OF NEW INSPECTION INTERVAL TO EARLIER INSPECTION INTERVAL VS. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION. THE RATIO ENSURES THAT WITH A 97% CONFIDENCE NO MORE THAN 3% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME	11
7-1C.	RATIO OF NEW INSPECTION INTERVAL TO EARLIER INSPECTION INTERVAL VS. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION. THE RATIO ENSURES THAT WITH A 98% CONFIDENCE NO MORE THAN 2% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME	12
7-1D.	RATIO OF NEW INSPECTION INTERVAL TO EARLIER INSPECTION INTERVAL VS. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION. THE RATIO ENSURES THAT WITH A 99% CONFIDENCE NO MORE THAN 1% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME	13
7-2A.	PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION VS. POPULATION SIZE FOR AN ALLOWABLE RANGE OF 1/8 IN THE RATIO OF THE NEW INSPECTION INTERVAL TO THE EARLIER INSPECTION INTERVAL BEFORE THE INSPECTION INTERVAL IS REQUIRED TO CHANGE. THE PERCENTAGE IS ESTIMATED WITH A 95% CONFIDENCE THAT NO MORE THAN 5% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME	14
7-2B.	PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION VS. POPULATION SIZE FOR AN ALLOWABLE RANGE OF 1/6 IN THE RATIO OF THE NEW INSPECTION INTERVAL TO THE EARLIER INSPECTION INTERVAL BEFORE THE INSPECTION INTERVAL IS REQUIRED TO CHANGE. THE PERCENTAGE IS ESTIMATED WITH A 95% CONFIDENCE THAT NO MORE THAN 5% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME	15

LIST OF ILLUSTRATIONS (Cont.)

Figure **Page**

7-2C. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION VS. POPULATION SIZE FOR AN ALLOWABLE RANGE OF 1/8 IN THE RATIO OF THE NEW INSPECTION INTERVAL TO THE EARLIER INSPECTION INTERVAL BEFORE THE INSPECTION INTERVAL IS REQUIRED TO CHANGE. THE PERCENTAGE IS ESTIMATED WITH A 97% CONFIDENCE THAT NO MORE THAN 3% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME 16

7-2D. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION VS. POPULATION SIZE FOR AN ALLOWABLE RANGE OF 1/4 IN THE RATIO OF THE NEW INSPECTION INTERVAL TO THE EARLIER INSPECTION INTERVAL BEFORE THE INSPECTION INTERVAL IS REQUIRED TO CHANGE. THE PERCENTAGE IS ESTIMATED WITH A 98% CONFIDENCE THAT NO MORE THAN 2% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME 17

7-2E. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION VS. POPULATION SIZE FOR AN ALLOWABLE RANGE OF 1/3 IN THE RATIO OF THE NEW INSPECTION INTERVAL TO THE EARLIER INSPECTION INTERVAL BEFORE THE INSPECTION INTERVAL IS REQUIRED TO CHANGE. THE PERCENTAGE IS ESTIMATED WITH A 95% CONFIDENCE THAT NO MORE THAN 1% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME 18

7-2F. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION VS. POPULATION SIZE FOR AN ALLOWABLE RANGE OF 1/6 IN THE RATIO OF THE NEW INSPECTION INTERVAL TO THE EARLIER INSPECTION INTERVAL BEFORE THE INSPECTION INTERVAL IS REQUIRED TO CHANGE. THE PERCENTAGE IS ESTIMATED WITH A 95% CONFIDENCE THAT NO MORE THAN 3% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME 19

7-2G. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION VS. POPULATION SIZE FOR AN ALLOWABLE RANGE OF 1/4 IN THE RATIO OF THE NEW INSPECTION INTERVAL TO THE EARLIER INSPECTION INTERVAL BEFORE THE INSPECTION INTERVAL IS REQUIRED TO CHANGE. THE PERCENTAGE IS ESTIMATED WITH A 95% CONFIDENCE THAT NO MORE THAN 2% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME 20

LIST OF ILLUSTRATIONS (Cont.)

Figure **Page**

7-2H. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION VS. POPULATION SIZE FOR AN ALLOWABLE RANGE OF 1/3 IN THE RATIO OF THE NEW INSPECTION INTERVAL TO THE EARLIER INSPECTION INTERVAL BEFORE THE INSPECTION INTERVAL IS REQUIRED TO CHANGE. THE PERCENTAGE IS ESTIMATED WITH A 95% CONFIDENCE THAT NO MORE THAN 1% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME 21

8-1. RATIO OF NEW INSPECTION INTERVAL TO EARLIER INSPECTION INTERVAL VS. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION. THE RATIO ENSURES THAT WITH 95% CONFIDENCE NO MORE THAN 5% OF THE IN-SERVICE POPULATION OF 12,000 UNITS IS FAILED AT ANY GIVEN TIME 24

8-2. RESPONSE OF THE PERCENTAGE OF LIGHT BULBS FOUND FAILED ON INSPECTION AND THE UPPER PREDICTION LIMIT TO A DECREASE IN THE INSPECTION INTERVAL FROM 12 MONTHS TO 7 MONTHS 27

C-1. NORMAL PROBABILITY DISTRIBUTION C-3

C-2. QUICK BASIC PROGRAM FOR CALCULATING K OR F_U C-7

LIST OF TABLES

Table		Page
8-1.	DEFECTIVE BULBS FOUND AT MONTHLY INSPECTIONS IN A FIXED POPULATION OF 12,000 BULBS	23
D-1.	SIMULATION RESULTS	D-4
D-2.	SIMULATION RESULTS	D-6
E-1.	SIMULATION RESULTS	E-2
E-2.	SIMULATION RESULTS	E-3
E-3.	SIMULATION RESULTS	E-4
E-4.	SIMULATION RESULTS	E-5
E-5.	SIMULATION RESULTS	E-6
E-6.	SIMULATION RESULTS	E-7

EXECUTIVE SUMMARY

Procedures have been developed to establish inspection intervals for a population of regularly maintained units that, in use, can fail at random times. These procedures are intended to attain performance goals to ensure that no more than a certain small percentage of the units will be failed at any one time. Methods to choose and to change the length of time of the inspection interval are specified. Phase-in, i.e., the transition from one inspection interval to another inspection interval and the transition to a performance goal, is described.

A formula has been developed to predict, on a statistical basis, the upper limit of the percentage of the population that will be in a failed state during a future cycle based upon the number of defective units found in the previous inspection cycle. To estimate how well the formula will perform, repeated simulations of the upper limit using randomly generated failure data were compared with repeated simulations of independently generated random numbers of units found failed at inspection. The formula is primarily for use when the population is greater than 100 units and the percentage of the population found failed at inspection is greater than 1%.

To illustrate these procedures, the formula has been applied to the maintenance of a population of light bulbs in a building where bulb failures are not to exceed a given small percentage of the bulb population. The formula can apply equally to fleets of trucks or taxis, or to populations of rail cars or transit cars.

PROCEDURES TO ESTABLISH INSPECTION INTERVALS

1. BACKGROUND

Along with the assurance of a long, reliable life, in-service equipment may be required to operate safely and economically with some failed units. Procedures for inspection, repair, and replacement can be employed to permit operation with no more than a specified small percentage of failures.

A statistical procedure of systematic inspections may be used to limit the percentage of the population that is failed at a given point in time when the units of the population are inspected and repaired at regular intervals. The procedure permits this percentage to be estimated with a sequence of regular inspections conducted over a given period. During this period, the entire population would be inspected at different times, thereby, enabling the time between inspections and repairs to be used to control the percentage of failures.

With some populations, as varied as fleets of trucks or populations of lamps in large buildings, an issue is ensuring that a given percentage of the population of units in service is functioning properly. The units can and do operate practically with a small percentage of failures. Establishing a statistical procedure for inspection, repair, and replacement can ensure that at a given point in time and with a high probability (e.g., 98%), no more than a certain percentage (e.g., 5%) of the units in service will be failed.

2. OBJECTIVE

The objective of this paper is to describe a statistical inspection tool relating the interval of time required to inspect a given population of regularly maintained units subject to failure to the percentage of the population that is failed at a given point in time.

3. PROBABILITY OF FAILURE OF A SINGLE UNIT

Periodic maintenance inspections are conducted on a population of identical working units that tend to fail in service. Except for those units that have just been inspected, the condition of each unit within a population is unknown. Thus, the time of failure cannot be determined precisely and is assumed to be random. Some small number of units is known to fail in the regular interval between inspections. At each inspection, failed units are repaired to their original condition and returned to service to maintain a fixed population size.

The intervals between successive failures are assumed to be independently and identically distributed. The start of each inspection of any unit is considered to be the start of a new lifetime of that unit with the same probability of failure in the interval between some time, t, and some subsequent time, $t+\tau$. This probability depends only on the length of the interval, τ, and not on the previous time, t. In any interval, $\tau \succ 0$, the probability of failure is greater than zero. The interval is assumed to be sufficiently small to permit, at most, only one failure to occur.

Each unit is assumed to have an exponentially distributed failure time. All units that are not failed, including repaired units, are assumed to fail at a constant average rate, λ. Further, since each unit of the population is identical and is inspected at the same regular inspection interval, T, the probability, P, that any unit, not failed at any time, t, will fail before some later time, t+T, is an exponential function,

$$\mathbf{P=1-e^{-\lambda T},}$$

that depends only on T.

4. INSPECTION SCHEDULE AND INSPECTION INTERVAL PHASE-IN

The population of N units is divided into M groups of equal size. An inspection interval T is established and maintained, and is the same for each group and for each unit within its group. If the number of groups is fairly large, the prescribed conditions for inspection should approximate a uniform regular inspection with interval T. Figure 4-1 illustrates the sequential division of time for inspecting all units of the population.

In the first stage of the inspection procedure, every unit is inspected and repaired as needed according to a proposed interval, T_1, although repairs to these units are not recorded at this time. In the second stage, every unit is inspected and repaired as needed, and data related to all repairs are recorded. Second stage inspections will be repeated in the inspection interval, T_1, until data indicate a need to establish a new inspection interval, T_2.

The transition to a new interval starts at the beginning of the third stage with the selection of the first unit in the first group to be inspected. The procedure calls for immediate change to the new rate of inspection while continuing the same cyclic order of inspecting units of the population. The percentage of the population found failed at inspection and the corresonding percentage of the population that is failed at a given point in time, change with each monthly inspection until the new interval, T_2, is established (at the end of T_2 units of time) when the entire population has been inspected.

In the fourth stage, the percentage of units found failed at inspection settles, within a subsequent T_2 units of time, to a corresponding percentage of the population that is failed at a given point in time.

Thus, the entire process of phase-in comprising the transition to the percentage of the population that is failed at a given point in time at the new inspection interval, T_2, occurs within two T_2 units of time.

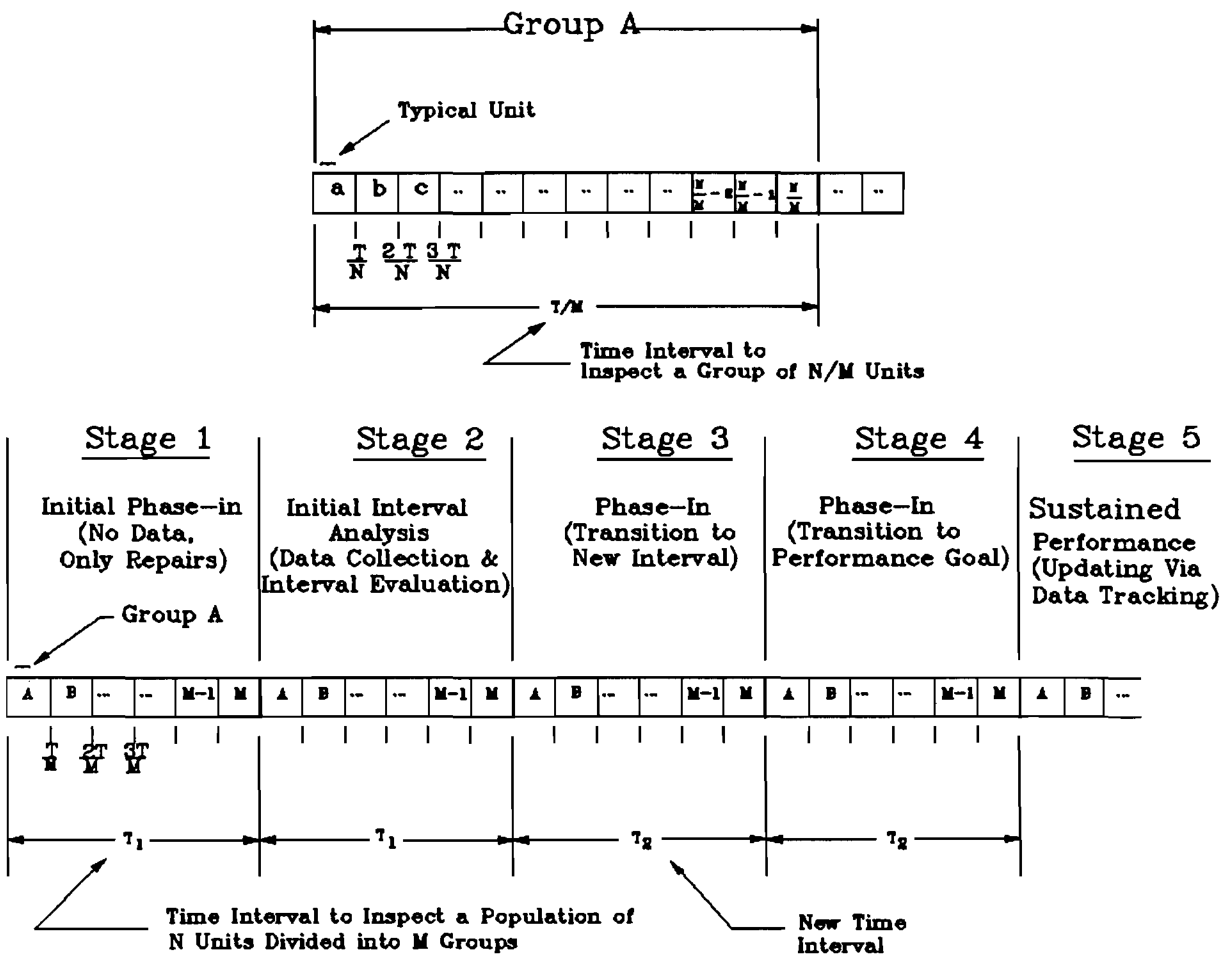


FIGURE 4-1. STAGES IN AN INSPECTION SCHEDULE

5. EXPECTED VALUE OF FAILED UNITS AND THE UPPER PREDICTION LIMIT

When a regular inspection interval has been established, i.e., when the population has been inspected once in the first interval, T_1, and all units have been repaired upon inspection where necessary, there are two measurable quantities that are relatable to the failure rate, λ: the percentage of the population found failed at inspection, R, and the percentage of the population that is failed at a given point in time, F. Data from each inspection show the number of that part of the population that was failed at the time of inspection and required repairs. (See Table 8-1.) The proportion of the population found failed at inspection in the interval, T, is determined by dividing the total number of units found failed on inspection, D, by the number of units in the population, N. Hence:

$$R = D/N.$$

The percentage of the population that is failed, F, at a given point in time, however, is unknown. Its relationship with the inspection interval must be determined. Then, changes in the inspection interval can be used to control the percentage of the population that is failed at a given point in time.

Using a probabilistic approach, E(R) and E(F), respectively, the expected values of the percentage of the population found failed at inspection, R, and the percentage of the population that is failed at a given point in time, F, can be related to the failure rate, λ, and to the inspection interval, T. R and F are random variables. E(R) and E(F), average values that are representative of some distribution of values, are the fixed but not known expected values. If each distribution remains the same, the expected value of each random variable would be observed again and again very many times. In that case, the portion of the population expected to be found failed at inspection is

$$E(R) = 1 - e^{-\lambda T_1}$$

and the portion of the population expected to be failed at a given point in time is

$$E(F)=1-\frac{1}{\lambda T_2}(1-e^{-\lambda T_2}).$$

(A more detailed description is found in Appendices A and B.) The quantities are found for different inspection intervals, T_1 and T_2. The case where the percentage of the population found failed at inspection, R, and the percentage of the population that is failed at a given point in time, F, refers to different inspection intervals that will be of particular interest. The two equations can be combined, eliminating the failure rate, λ, so that the expected portion of the population in a failed state at any given time is

$$E(F)=1+\frac{1-(1-E(R)^K)}{K\ \log_e(1-E(R))}$$

where K is the ratio of the new inspection interval to the previous inspection interval:

$$K=T_2/T_1.$$

This expression is used to relate the expected value of the percentage of the population in a failed state at a given point in time, E(F), at the new interval, T_2, to the expected value of the percentage of the population found failed at inspection, E(R), at the earlier interval, T_1, and to the ratio of the new interval to the earlier interval, K.

It is important to remember that E(R) and E(F) are abstract quantities and have to be related to quantities that are real and measurable. In much of the remainder of this report, that relationship is developed in terms of R and F, which are real and measurable. Since E(R) and E(F) are average values of some distributions, a limit which will be exceeded by the percentage of the population that is failed at a given point in time, F, in no more than some small percentage of all possible cases, must also be established. This limit, F_U, sometimes called the upper confidence or upper prediction limit is a function of the population size, N, the percentage of the population found failed at inspection, R, and the probability that F is greater than F_U. The relationship, in effect, says, that F will exceed F_U (e.g., 5%) in no more than α (e.g., 2%) of all cases and is equivalent to being (1-α) (e.g., 98%) confident that a given quantity of units in service will not be in a failed state.

The derivation of the formula for F_U is described in Appendix A. The derivations of some of the variances used in Appendix A are described in Appendix B. The formulas to determine the upper prediction limit and the new inspection interval along with the code for performing the calculations and a few simple examples are described in Appendix C. The simulation testing developed to study the performance of the formula for F_U is described in Appendix D. The results of several simulation test runs appear in Appendix E. A possible approach to choose and update the inspection interval is suggested in Appendix F. Additional comments on the use of the formulas are given in Appendix G.

Note: When the expected value of the percentage of the population found failed at inspection, E(R), is very small and the ratio of the current inspection interval to the previous inspection interval, K, is moderate, an approximation of the expected value of the portion of the population that is failed at a given point in time is:

$$E(F) \approx \frac{K}{2} E(R).$$

If a measured percentage of the population found failed at inspection, R, is substituted for the expected value, E(R), in the above equation, the percentage of the population that is failed at a given point in time, F, can be estimated and can provide statistical bounds for F based on the percentage of the population found failed at inspection, R, usually, for a previous inspection interval before the given time for F. (Graphical results for the upper prediction limit in Section 7 when compared with the above equation evaluated in terms of R show close agreement for a large population and deterioration as the population decreases. This equation is offered only to show the approximate dependence involved and can be quite inaccurate in many cases. Much more accurate estimates are discussed in Appendix A.)

6. SIMULATION TESTING

Determination of an exact formula for the upper prediction limit, F_U, specified in terms of the population size, N, the percentage of the population found failed at inspection, R, the initial inspection interval, T_1, the new inspection interval, T_2, and the probability, α, that the upper prediction limit will be exceeded by the percentage of the population that is failed at a given point in time appears to be beyond the capability of current statistical techniques. However, an asymptotic formula for F_U has been derived that works well when the number of failures in the entire population, NR, is large. In this case, both the percentage of the population found failed at inspection, R, and the percentage of the population that is failed at a given point in time, F, tend to be normally distributed. (See Appendix A).

Though probability theory is useful for determining the form of the function for the upper prediction limit, F_U, when the number of units in the population found failed at inspection, NR, is large, the theory is not capable of accurately determining the upper prediction limit when the number of units in the population found failed at inspection is not large. Thus, repeated simulations for determining the upper prediction limit in terms of population size, the percentage of the population found failed at inspection, the current and new inspection intervals, and the probability that the upper prediction limit will be exceeded, have been used to predict the performance of the proposed asymptotic formula. Each simulation study (which generates separate data on each unit of the population) is repeated many times (10,000 to 200,000) at each choice of the parameters.

Results of the simulation studies enable comparison between the excess of the percentage of the population that is failed at a given point in time, F, greater than the upper prediction limit, F_U, for many representative choices of the other parameters and the probability, α, that F is greater than F_U. Simulations can show the amount that F_U would have to be increased in order to limit the exceedances to α of all cases when F_U is not large enough. Conversely, they can also show how much the proposed value for F_U could be decreased and still limit exceedances to the desired level when F_U is too large.

The results of repeated simulation studies discussed in Appendix D show that as long as the expected value of the percentage of the population found failed at inspection is not less than 1% and the population size is not less than 100, the formula appears to be satisfactory, being too conservative in some cases and not conservative enough in others. The formula usually errs on the conservative side and usually differs only slightly from the true upper prediction limit, F_U, which is exceeded in exactly α of all cases.

7. TYPICAL RESULTS OF THE ANALYSIS

Graphical evaluations of the formula for the upper prediction limit are presented in two different figures. Individual curves are plotted in both figures for a given upper prediction limit and confidence level.

The curves in Figures 7-1A to 7-1D, each representative of a given population size, show how the time interval ratio is related to the percentage of the population found failed at inspection. For the hypothetical maintenance problem, the 12,000 unit population curve of Figure 7-1A, redrawn as Figure 8-1, is used in the determinations of the inspection interval required to limit light bulb failures and the allowable variation in the percentage of bulbs found failed at inspection before the inspection interval is required to be changed.

In Figures 7-2A to 7-2H, two curves, evaluated for different inspection interval ratios, are plotted to relate the percentage of units found failed at inspection to the population size. The inspection interval ratios are selected to establish the boundaries limiting the region at the allowable variation in the percentage of bulbs found failed at inspection in the hypothetical maintenance problem.

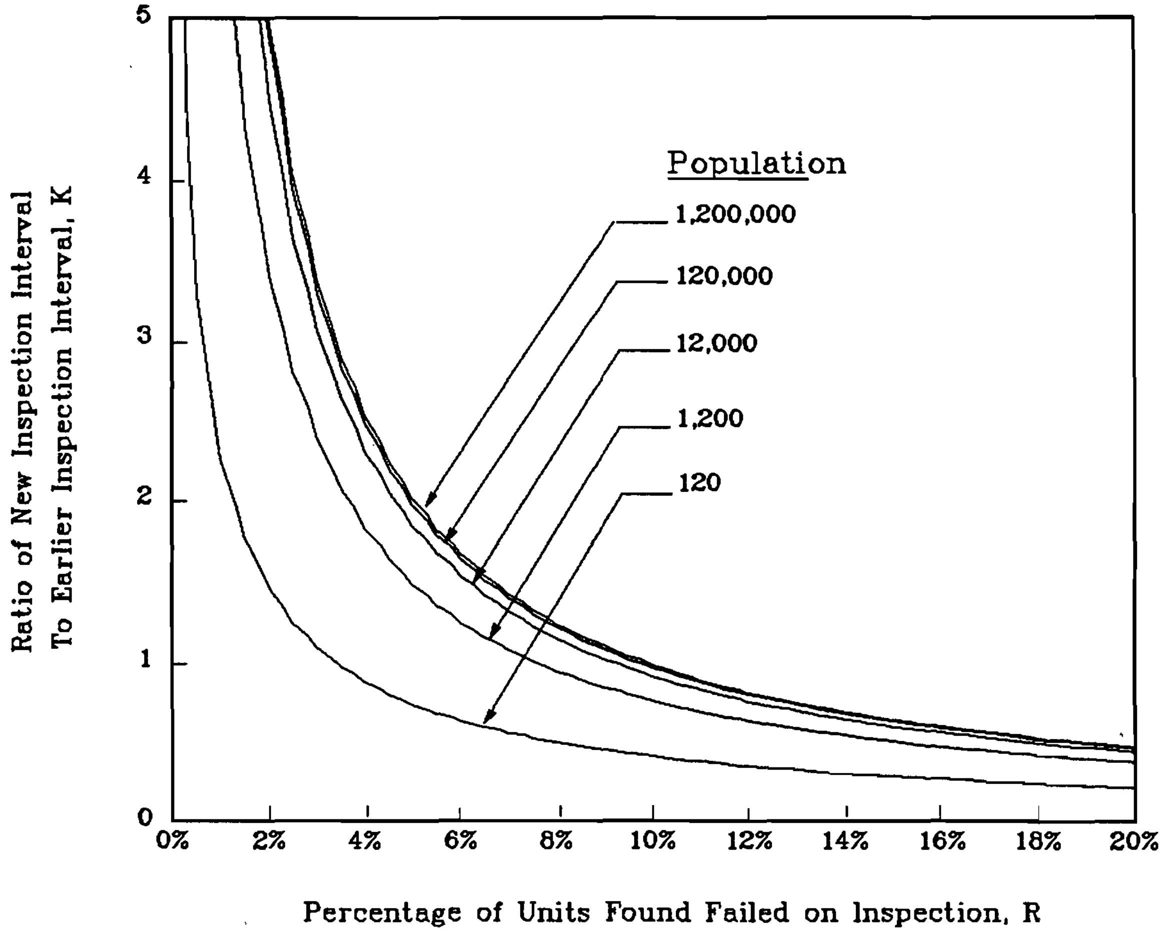


FIGURE 7-1A. RATIO OF NEW INSPECTION INTERVAL TO EARLIER INSPECTION INTERVAL VS. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION. THE RATIO ENSURES THAT WITH A 95% CONFIDENCE NO MORE THAN 5% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME.

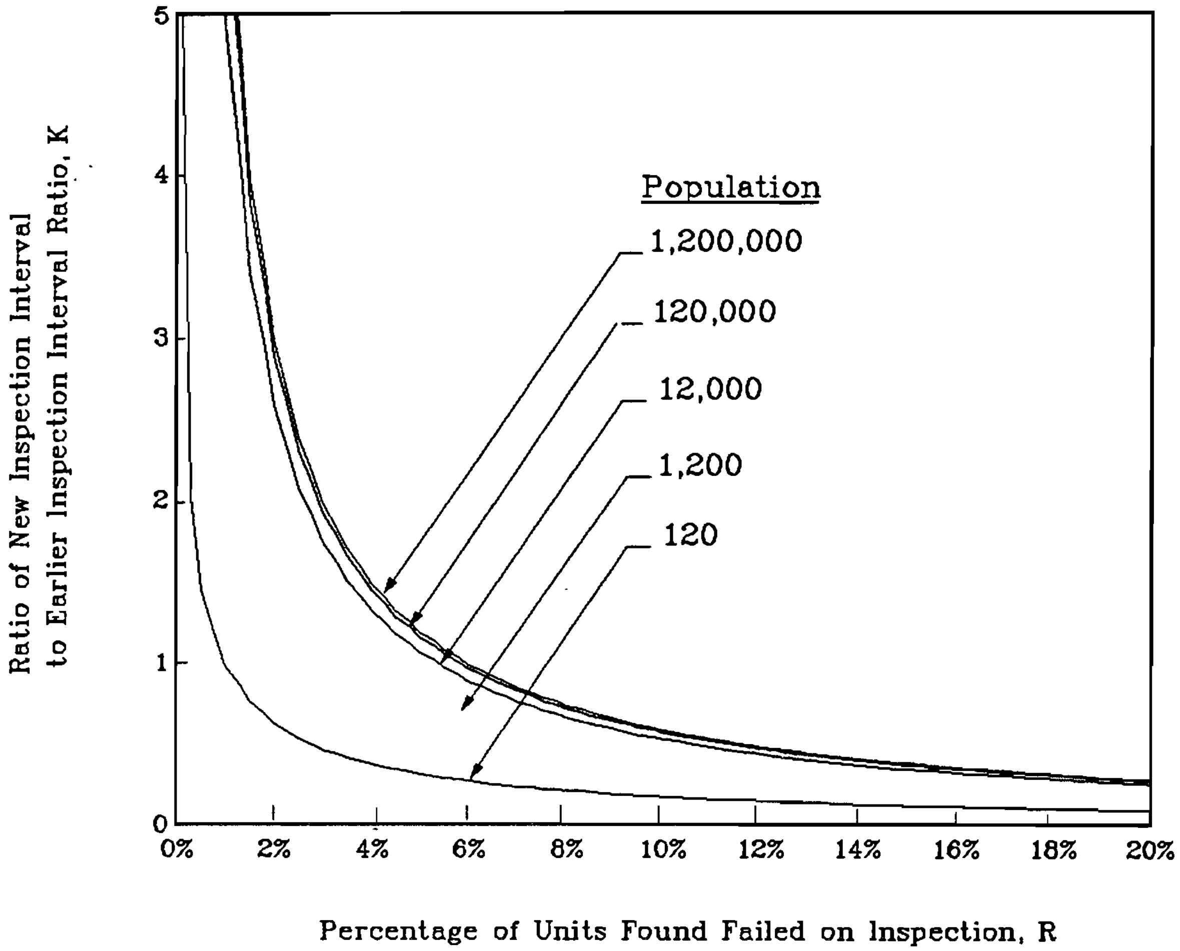


FIGURE 7-1B. RATIO OF NEW INSPECTION INTERVAL TO EARLIER INSPECTION INTERVAL VS. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION. THE RATIO ENSURES THAT WITH A 97% CONFIDENCE NO MORE THAN 3% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME.

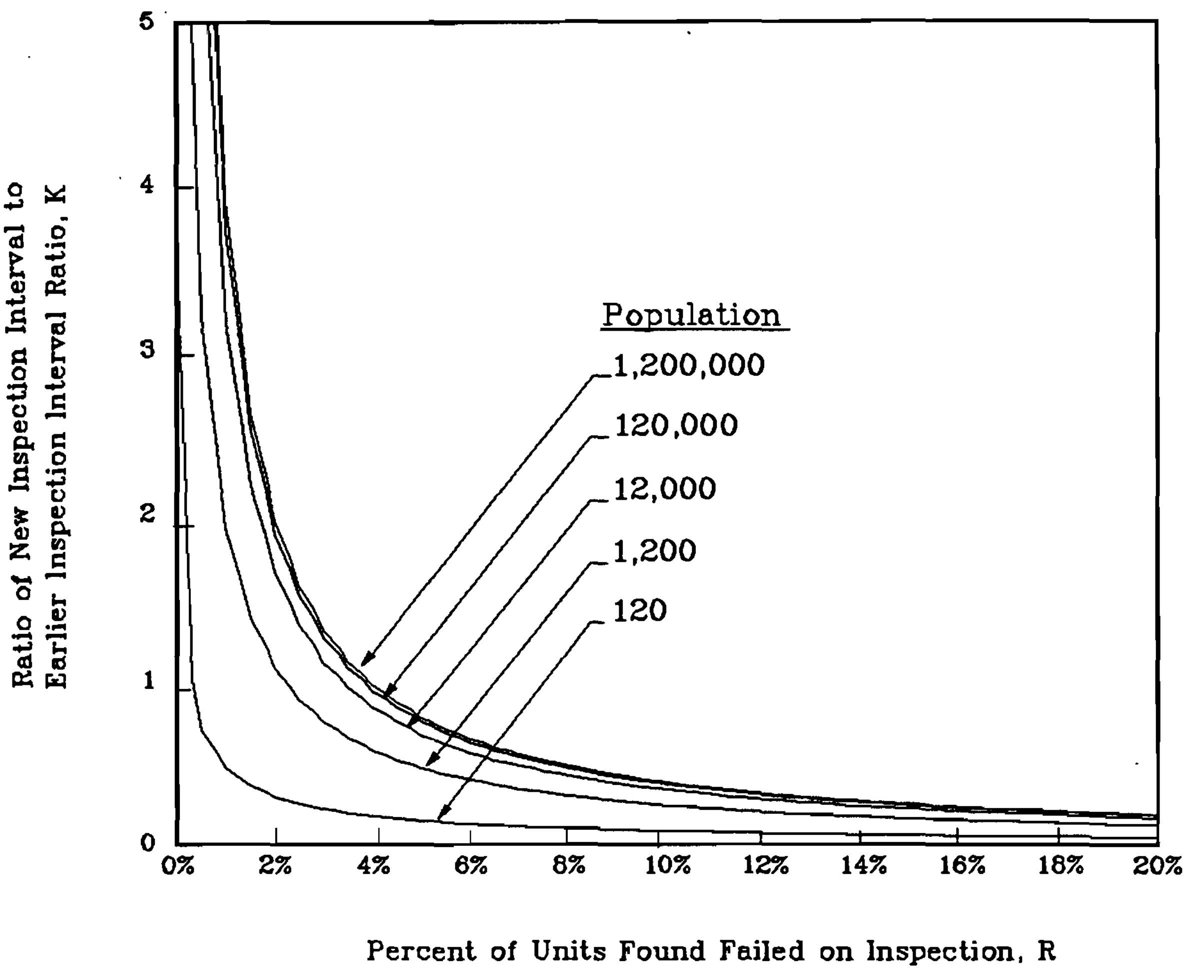


FIGURE 7-1C. RATIO OF NEW INSPECTION INTERVAL TO EARLIER INSPECTION INTERVAL VS. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION. THE RATIO ENSURES THAT WITH A 98% CONFIDENCE NO MORE THAN 2% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME.

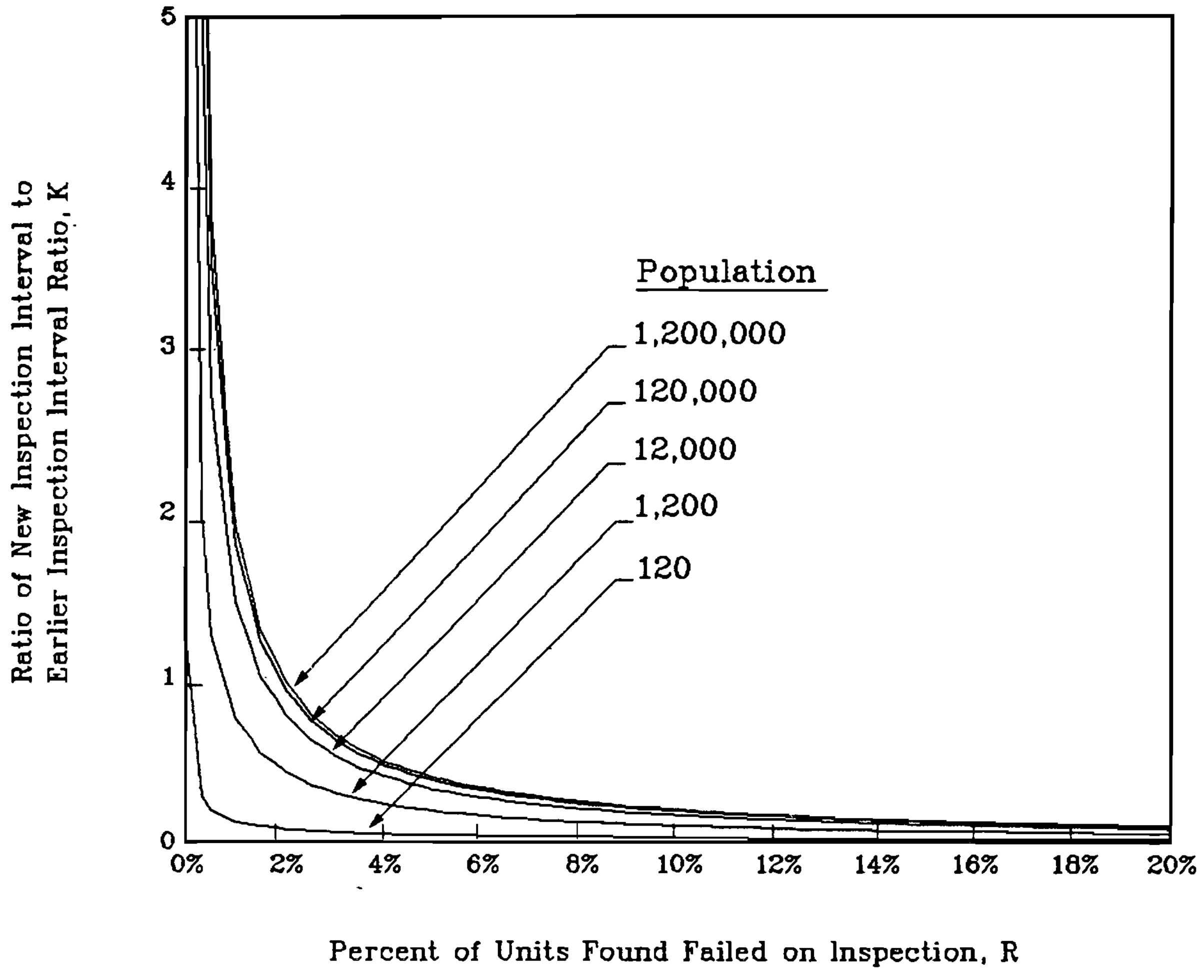


FIGURE 7-1D. RATIO OF NEW INSPECTION INTERVAL TO EARLIER INSPECTION INTERVAL VS. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION. THE RATIO ENSURES THAT WITH A 99% CONFIDENCE NO MORE THAN 1% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME.

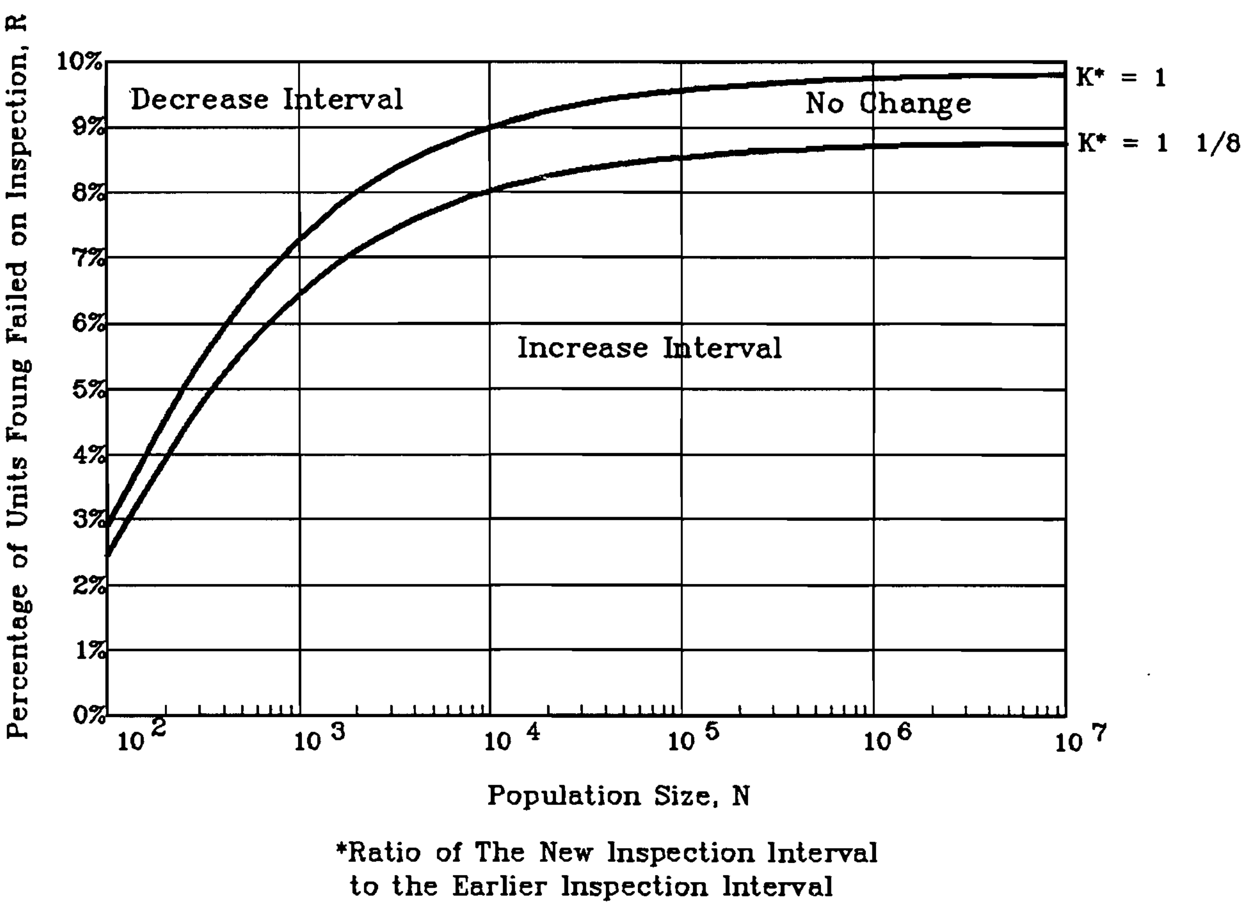


FIGURE 7-2A. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION VS. POPULATION SIZE FOR AN ALLOWABLE RANGE OF 1/8 IN THE RATIO OF THE NEW INSPECTION INTERVAL TO THE EARLIER INSPECTION INTERVAL BEFORE THE INSPECTION INTERVAL IS REQUIRED TO CHANGE. THE PERCENTAGE IS ESTIMATED WITH A 95% CONFIDENCE THAT NO MORE THAN 5% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME.

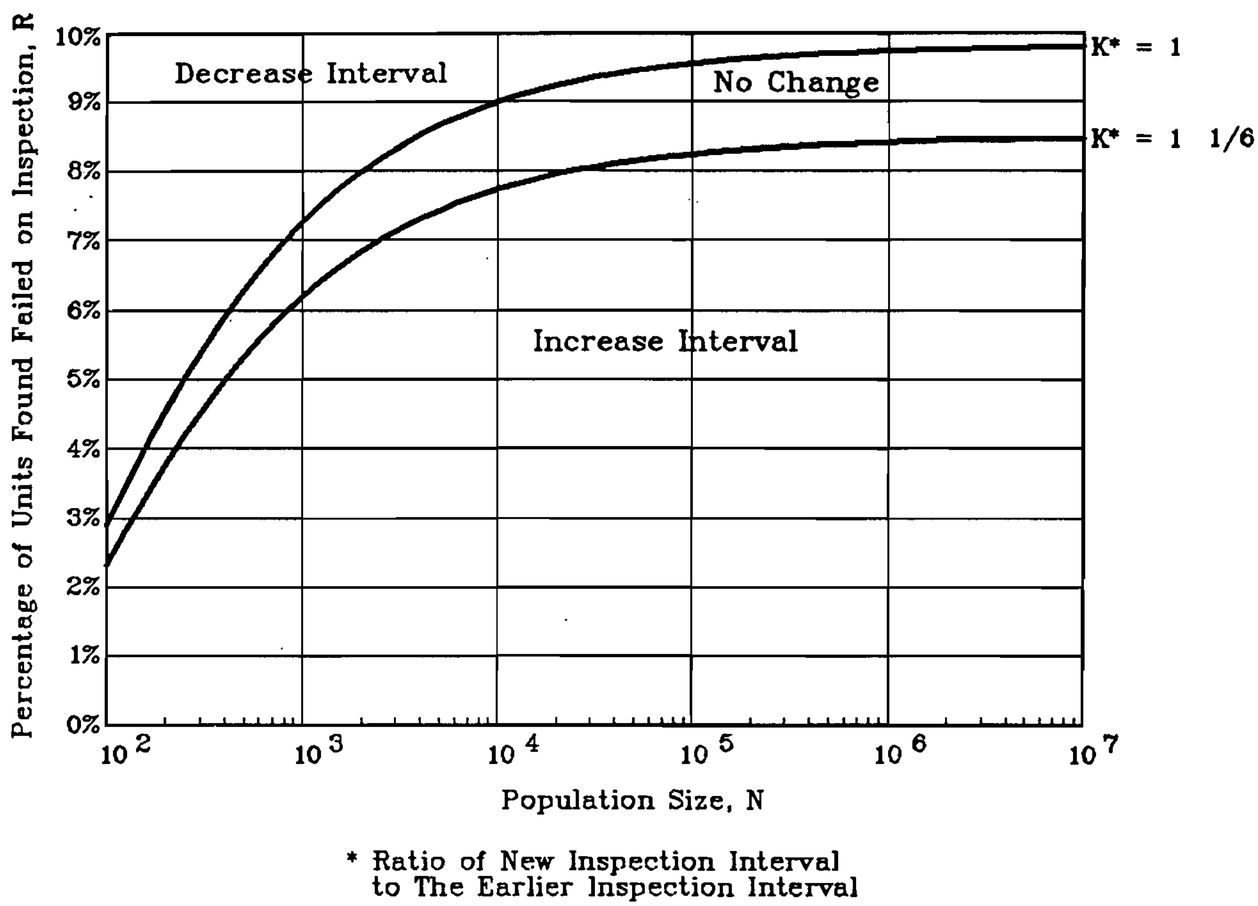


FIGURE 7-2B. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION VS. POPULATION SIZE FOR AN ALLOWABLE RANGE OF 1/6 IN THE RATIO OF THE NEW INSPECTION INTERVAL TO THE EARLIER INSPECTION INTERVAL BEFORE THE INSPECTION INTERVAL IS REQUIRED TO CHANGE. THE PERCENTAGE IS ESTIMATED WITH A 95% CONFIDENCE THAT NO MORE THAN 5% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME.

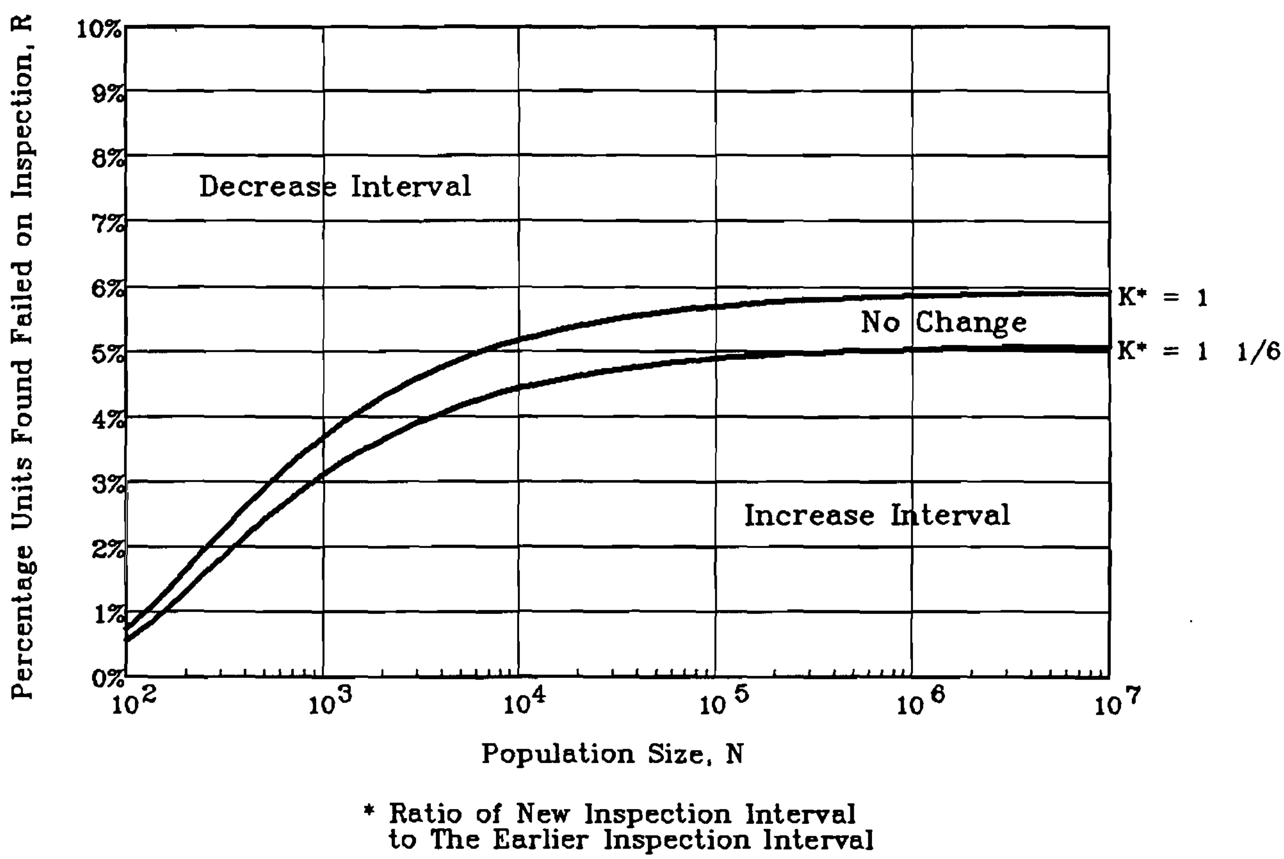


FIGURE 7-2C. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION VS. POPULATION SIZE FOR AN ALLOWABLE RANGE AT 1/8 IN THE RATIO OF THE NEW INSPECTION INTERVAL TO THE EARLIER INSPECTION INTERVAL BEFORE THE INSPECTION INTERVAL IS REQUIRED TO CHANGE. THE PERCENTAGE IS ESTIMATED WITH A 97% CONFIDENCE THAT NO MORE THAN 3% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME.

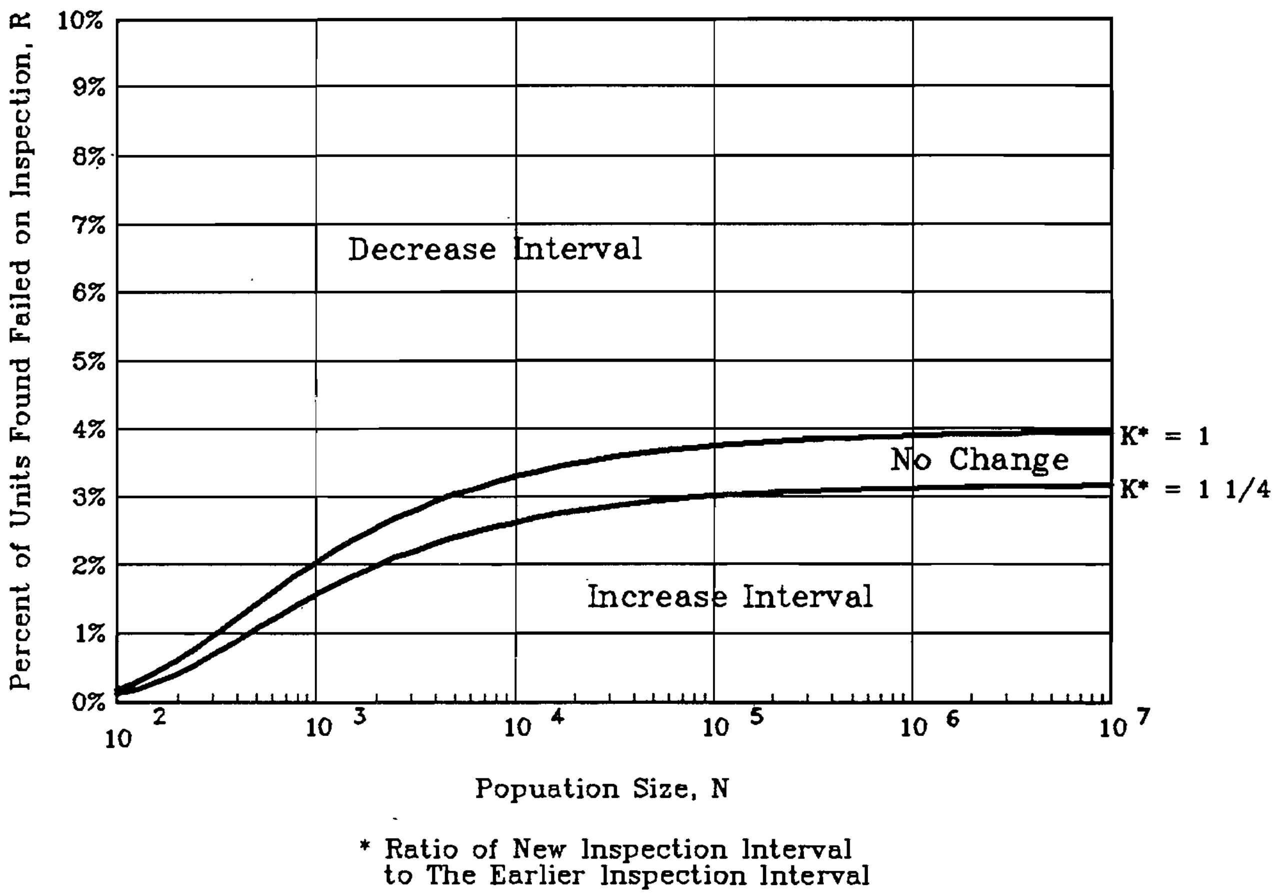
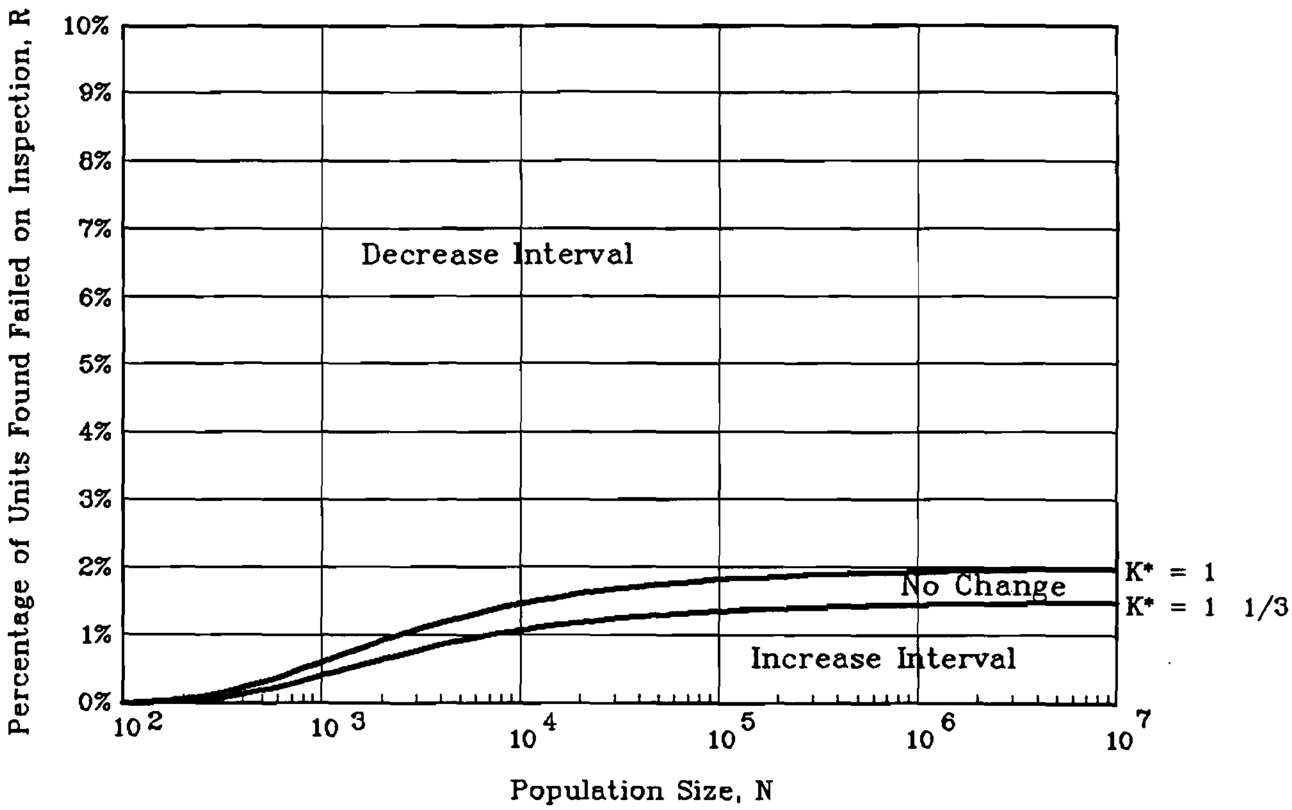


FIGURE 7-2D. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION VS. POPULATION SIZE FOR AN ALLOWABLE RANGE OF 1/4 IN THE RATIO OF THE NEW INSPECTION INTERVAL TO THE EARLIER INSPECTION INTERVAL BEFORE THE INSPECTION INTERVAL IS REQUIRED TO CHANGE. THE PERCENTAGE IS ESTIMATED WITH A 98% CONFIDENCE THAT NO MORE THAN 2% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME.



* Ratio of New Inspection Interval
to The Earlier Inspection Interval

FIGURE 7-2E. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION VS. POPULATION SIZE FOR AN ALLOWABLE RANGE OF 1/3 IN THE RATIO OF THE NEW INSPECTION INTERVAL TO THE EARLIER INSPECTION INTERVAL BEFORE THE INSPECTION INTERVAL IS REQUIRED TO CHANGE. THE PERCENTAGE IS ESTIMATED WITH A 95% CONFIDENCE THAT NO MORE THAN 1% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME.

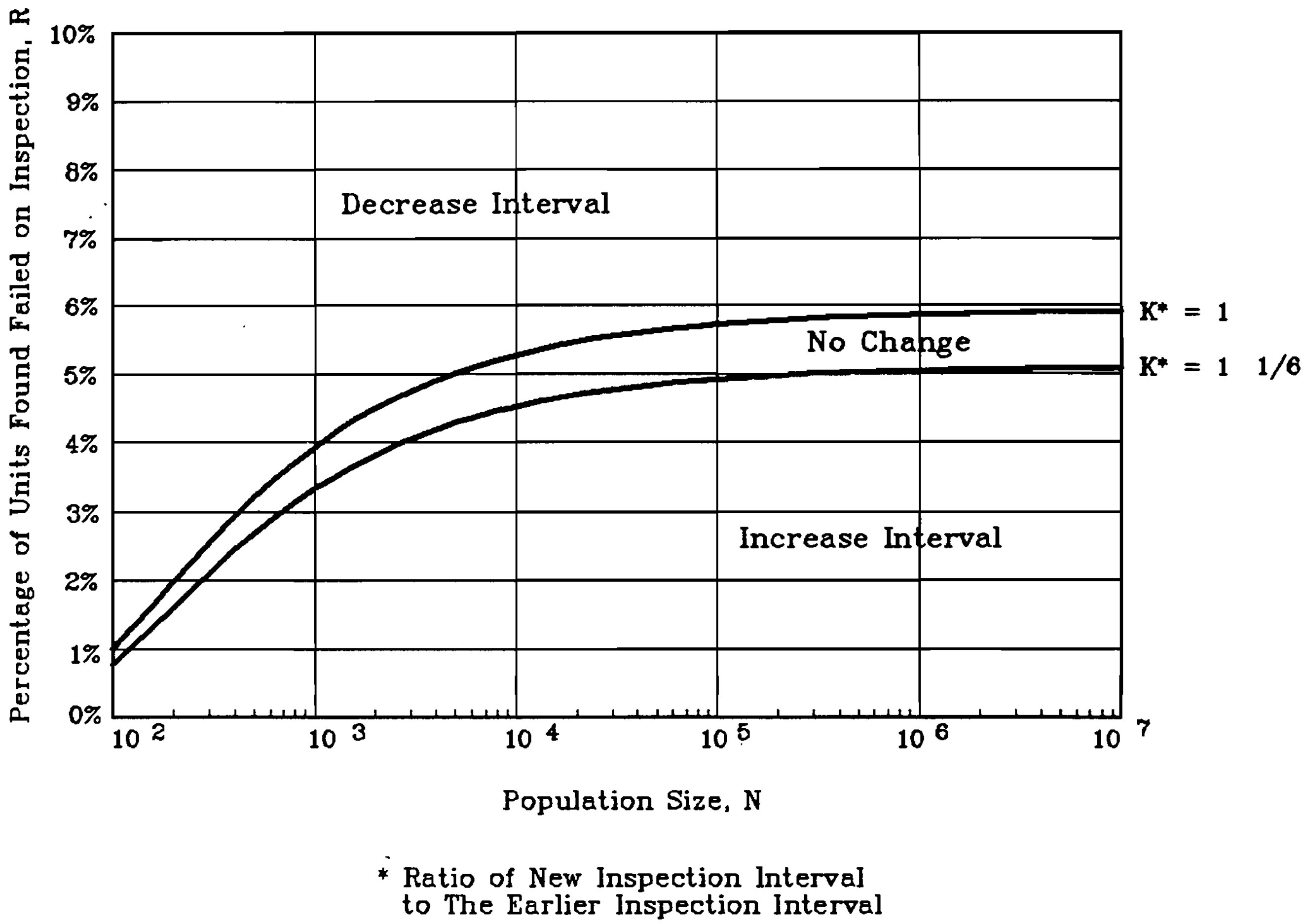


FIGURE 7-2F. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION VS. POPULATION SIZE FOR AN ALLOWABLE RANGE OF 1/6 IN THE RATIO OF THE NEW INSPECTION INTERVAL TO THE EARLIER INSPECTION INTERVAL BEFORE THE INSPECTION INTERVAL IS REQUIRED TO CHANGE. THE PERCENTAGE IS ESTIMATED WITH A 95% CONFIDENCE THAT NO MORE THAN 3% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME.

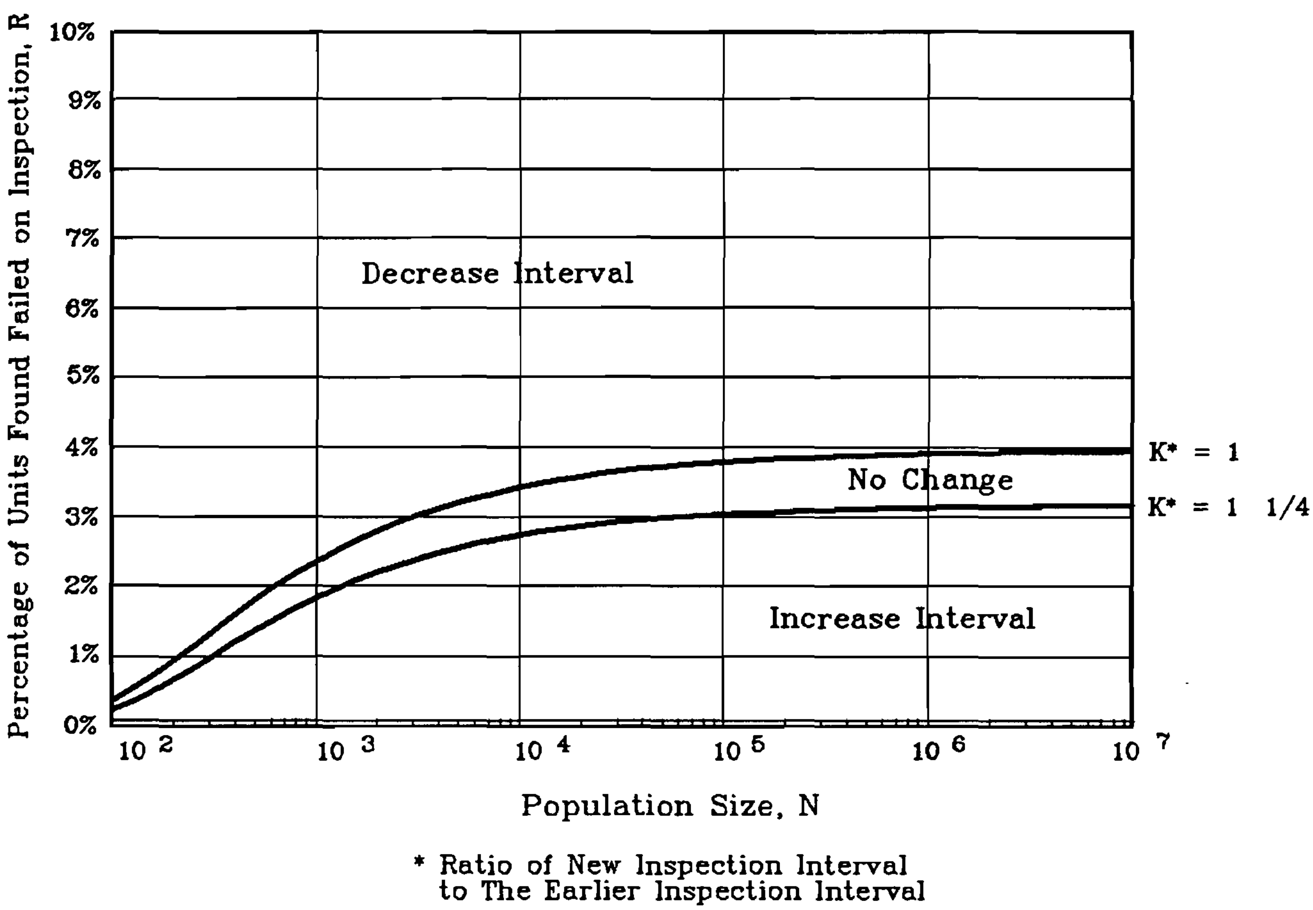


FIGURE 7-2G. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION VS. POPULATION SIZE FOR AN ALLOWABLE RANGE OF 1/4 IN THE RATIO OF THE NEW INSPECTION INTERVAL TO THE EARLIER INSPECTION INTERVAL BEFORE THE INSPECTION INTERVAL IS REQUIRED TO CHANGE. THE PERCENTAGE IS ESTIMATED WITH A 95% CONFIDENCE THAT NO MORE THAN 2% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME.

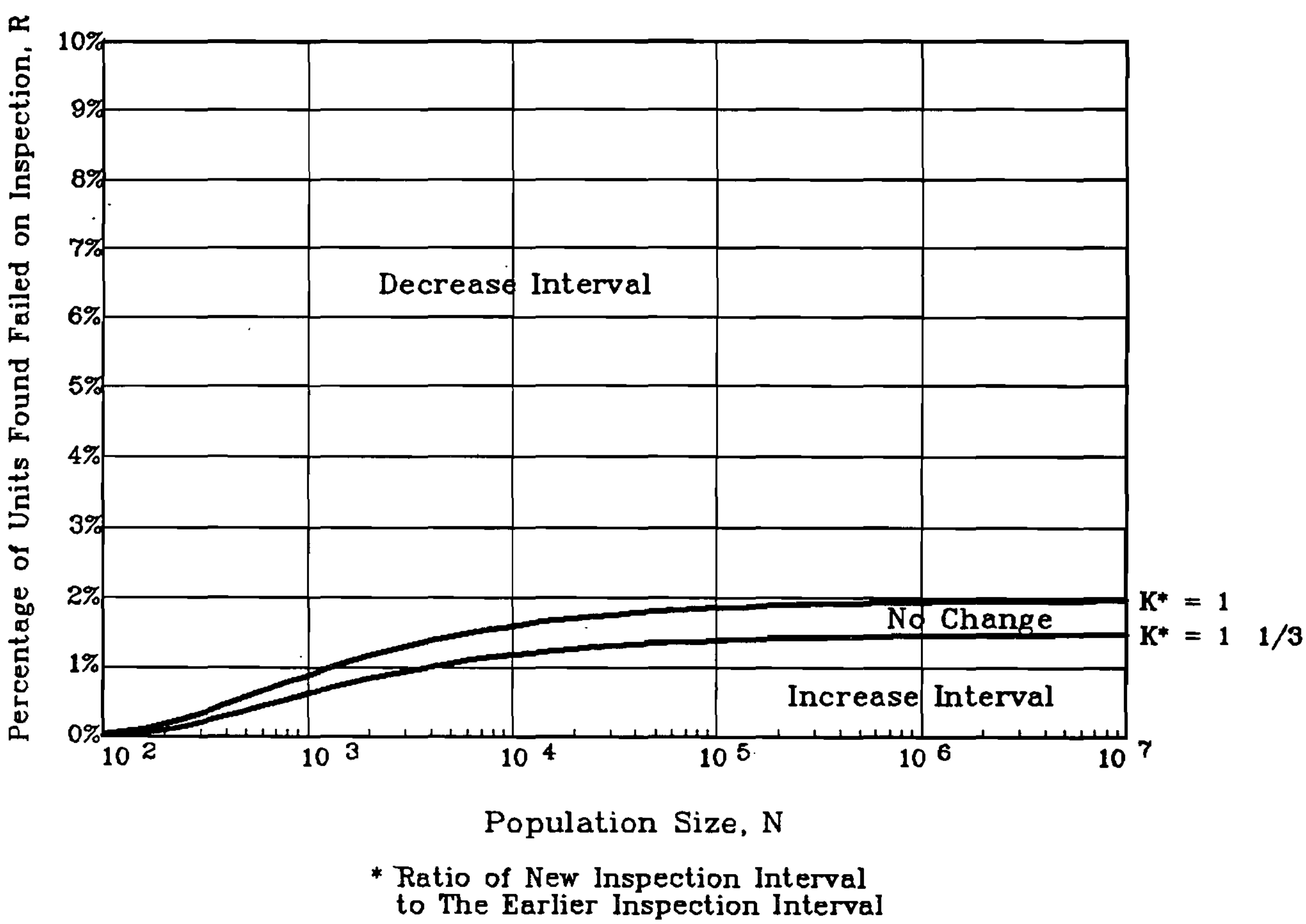


FIGURE 7-2H. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION VS. POPULATION SIZE FOR AN ALLOWABLE RANGE OF 1/3 IN THE RATIO OF THE NEW INSPECTION INTERVAL TO THE EARLIER INSPECTION INTERVAL BEFORE THE INSPECTION INTERVAL IS REQUIRED TO CHANGE. THE PERCENTAGE IS ESTIMATED WITH A 95% CONFIDENCE THAT NO MORE THAN 1% OF THE IN-SERVICE POPULATION IS FAILED AT ANY GIVEN TIME.

8. SOLUTION RESULTS APPLIED TO A HYPOTHETICAL MAINTENANCE PROBLEM

A 12-story building illuminated with 1,000 light bulbs on each floor serves as an example of the application of the statistical procedure of inspection and repair to maintain a given percentage of the bulbs in service. The goal is to permit, with 95% confidence, that no more than 5% of the bulbs are to be failed at a given point in time.

In Stage 1, all 12,000 light bulbs are divided into 12 groups. The routine calls for all of the bulbs to be inspected in a given regular uniform, floor- by-floor, sequence. A different group is inspected each month during the 12-month inspection interval. Bulbs found failed are replaced, but the number of bulbs found failed is not recorded. The amount of time to inspect all of the bulbs in the in-service population is the same as the inspection interval for each bulb.

In Stage 2, the number of failures in each group, recorded for the 12-month inspection interval, is shown in the first column of Table 8-1. The results are tallied at the end of each month and at the end of the inspection interval. Of the entire in-servive population of bulbs, 1800 bulbs (i.e., 15% of the population) are found failed and replaced.

According to Figure 8-1, the goal of 5% can be reached when the inspection interval is reduced by a factor equal to the inspection interval ratio, K=0.59. The new inspection interval (in months) is selected to be (T_2=KT_1=0.59 × 12=7.08 months) 7 months. When monthly inspections are conducted for this new inspection interval, the inspection interval ratio for K=1 (since inspections now are set to occur at a fixed interval) determines an upper bound of 9.1% for the percentage of bulbs found failed at inspection, R, that corresponds to 5% for the upper prediction limit of bulbs in a failed state at a given point in time.

The second column of Stage 2 shows the same number (1800) of failures rearranged for the 7 groups needed to inspect the bulbs in 7 months. The new group size consists of about 1714 bulbs which are located on the equivalent of 1 5/7 floors.

	Inspection Stage	
Inspection Month	1	2
1		141
2		138
3		152
4		126
5		158
6		139
7		163
8		161
9		157
10		174
11		137
12		154
TOTAL		1800

Table 8–1a. Failures Within 12 Groups of Equal Size for an Inspection Interval of 12 Months.

	Inspection Stage			
Inspection Month	2	3	4	5
1	240	249	152	153
2	245	226	164	147
3	250	203	152	158
4	259	181	145	163
5	274	163	147	158
6	280	149	164	165
7	252	137	132	172
TOTAL	1800	1308	1056	1116

Table 8–1b. Failures Within 7 Groups of Nearly Equal Size for An Inspection Interval of 7 Months.

TABLE 8-1. DEFECTIVE BULBS FOUND AT MONTHLY INSPECTIONS IN A FIXED POPULATION OF 12,000 BULBS

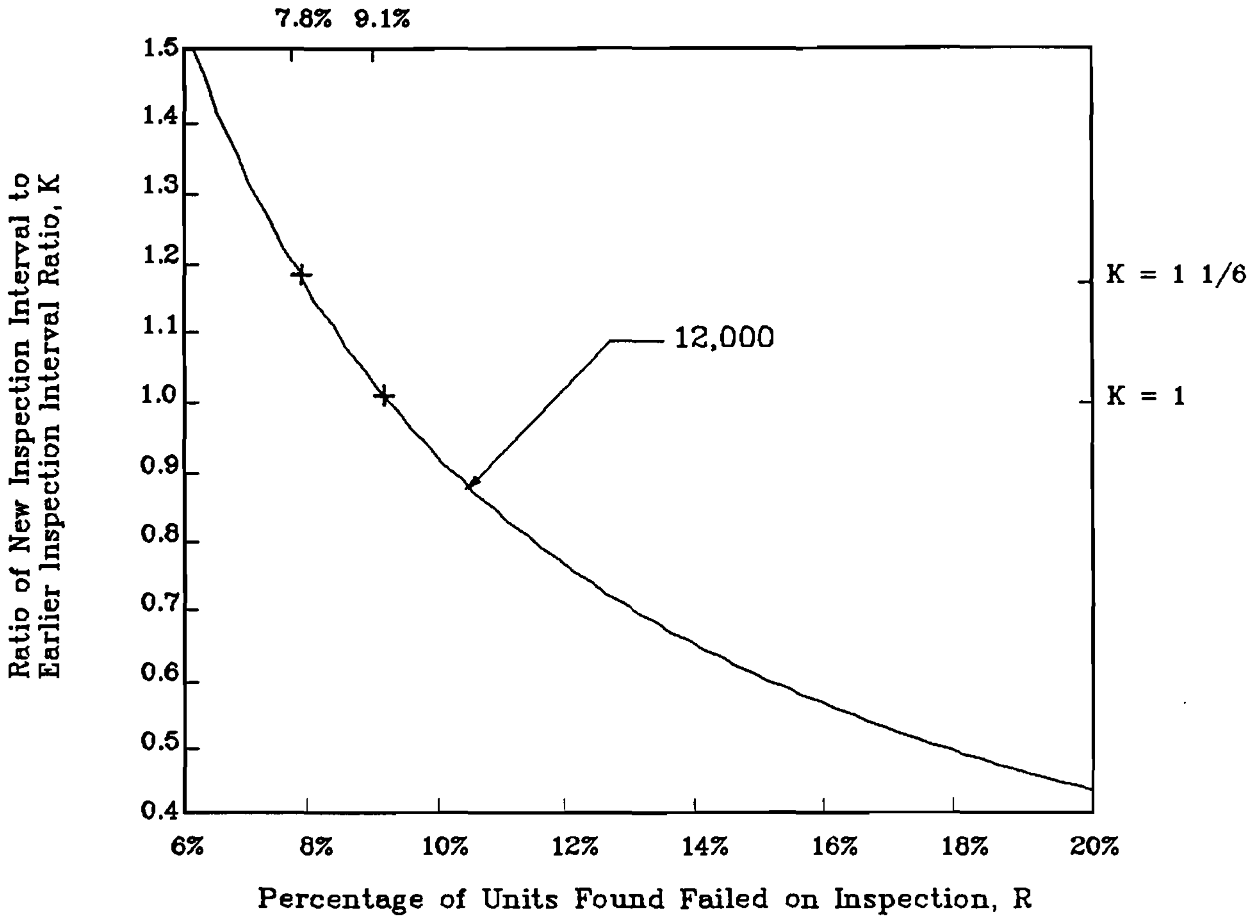


FIGURE 8-1. RATIO OF NEW INSPECTION INTERVAL TO EARLIER INSPECTION INTERVAL VS. PERCENTAGE OF UNITS FOUND FAILED ON INSPECTION. THE RATIO ENSURES THAT WITH 95% CONFIDENCE NO MORE THAN 5% OF THE IN-SERVICE POPULATION OF 12,000 UNITS IS FAILED AT ANY GIVEN TIME.

In Stage 3, the inspection interval changes from 12 months to 7 months. Inspections are conducted according to the new group size. At each monthly inspection, the time interval of inspection for bulbs not yet inspected in Stage 3 decreases by 5/7 month. At the end of the first monthly inspection, 1 5/7 floors are inspected while each of the previous 10 2/7 floors have not been inspected for 10 2/7 months representing an inspection interval for the entire population of bulbs of 11 2/7 months. Each successive monthly inspection represents another 5/7 month decrease in the interval until, at the end of 7 months, the 12-month inspection interval is reduced to 7 months.

In the meantime, the data collector needs to be aware of the change of the number and location of bulbs to be inspected in each new group. At the end of Stage 3, the data collected show that 10.9% of the bulbs were found failed at inspection.

During Stage 4, the percentage of bulbs found failed at inspection decreases to the upper bound of 9.1% (determined earlier) that corresponds to the upper prediction limit of 5%. At the end of stage 4, data indicate that 8.8% of inspected bulbs were found failed and did not exceed the upper bound or the upper prediction limit. Thus, the 5% goal was reached within 14 months of the change of the inspection interval from 12 months to 7 months. In subsequent inspections, the data collector continues to calculate the percentage of bulbs found failed at inspection in order to determine when an update is required and how the interval should be changed.

When the new inspection interval of 7 months was selected, the maintenance department also allowed the percentage of the bulbs found failed at inspection to vary by 1.3% before requiring an adjustment of the inspection interval. The upper bound on the percentage of the bulbs found failed at inspection for the 7-month inspection interval, as discussed earlier, was not to exceed 9.1%. Therefore, the percentage of bulbs found failed at inspection could vary between the upper bound of 9.1% and a lower bound of 7.8% without requiring the inspection interval to be changed. When the percentage of the bulbs found failed at inspection becomes less than the lower bound, the new inspection interval should be increased. According to Figure 8-1, the lower bound corresponds to an inspection interval that has increased in

9. LIMITATIONS

The procedure discussed in this paper is intended for use in adjusting inspection intervals in order to control the percentage of the population that is failed at a given point in time. The procedure tends to be more satisfactory for large populations and relatively less satisfactory for small populations for two reasons:

1. For very small populations of less than 100 units, the analytical techniques break down and fail to sustain the accuracy of the formula. The formula was developed using asymptotic techniques which are good for large populations and robust for intermediate populations. For very small populations, the formula ultimately fails to estimate the upper prediction limit of the percentage of the population that is failed at a given point in time.

2. Even if the formula were exact in its prediction of probabilities for very small populations, a small population size, still, requires more frequent inspections to overcome the loss of statistical precision in evaluating the percentage of the population that is failed at a given point in time and the high probability that this percentage not exceed the upper prediction limit by a certain amount.

(See also Apppendix G.)

APPENDIX A. DERIVATION OF THE FORMULA FOR F_U

The purpose of this section is to derive the formula for the upper prediction limit F_U for F. There are three major aspects to this derivation; a good point estimate or mean, a good prediction band based on the various sources of variation, and simulation testing of the prediction limit and level.

First, recall the meaning of F and of its prediction limit. F has meaning in a given population at a given time and represents the fraction of units that would be found failed if all units were to be examined simultaneously. Conversely, (1-F) is a measure of "availability." F is never measured, although it is an actualized or potentially measurable quantity. Instead, the fraction, R, found failed at inspection, is observed. In one inspection cycle of interval, T, every unit is inspected exactly once. All units are assumed to be inspected with the same delay, T, so that the units are always inspected in the same order and each unit, when it is inspected, was last inspected and repaired T units of time before. (The failure times are assumed independent and exponentially distributed. All units are assumed repaired and restored to initial conditions when found failed at inspection.)

Now, let us find a prediction limit for F in an inspection interval, T_2, based on the value of R observed in an earlier inspection interval, T_1. Then,

$$E(R)=1-e^{-\lambda T_1}$$

and,

$$E(F)=\frac{1}{T_2}\int_o^{T_2}(1-e^{-\lambda t})dt=1-\frac{1}{\lambda T_2}(1-e^{-\lambda T_2}).$$

(See Appendix B for more details.)

Combining the two equations,

$$E(F)=1+\frac{1-(1-E(R))^K}{K\ \log_e(1-E(R))}$$

where,

$$K=T_2/T_1.$$

That is to say E(F) is related to E(R) and K through the function F(x,y) by the equation:

$$E(F)=F(E(R),K)$$

where,

$$F(x,y)=1+\frac{1-(1-x)^y}{y\ \log_e(1-x)}.$$

Let σ_F denote the standard deviation in F, that is,

$$\sigma_F=\sqrt{E(F^2)-(E(F))^2}.$$

NF is actually the sum of N Bernoulli (zero-one) random variables, X_i, where $E(X_i)=p_i$. Then, the expected value of the number of units of the population in a failed state is,

$$N\ E(F)=\sum_{i=1}^{N}E(x_i)=\sum_{i=1}^{N}p_i$$

where,

$$p_i=1-e^{-\lambda t_i}$$

and, the expected value of the fraction of the population in a failed state is,

$$E(F)=\frac{\sum_{i=1}^{N} p_i}{N}\approx\frac{1}{T_2}\int_0^{T_2}(1-e^{-\lambda t})dt.$$

Since the X_i's are independent,

$$VAR[NF]=N^2\sigma_F^2=\sum_{i=1}^{N} p_i(1-p_i).$$

Thus,

$$\sigma_F^2=\sum_{i=1}^{N}\frac{(p_i-p_i^2)}{N^2}\approx\frac{1}{N\,T_2}\int_0^{T_2}(1-e^{-\lambda t})-(1-e^{-\lambda t})^2dt.$$

The result is

$$\sigma_F^2\approx\frac{(1-e^{-\lambda T_2})-.5(1-e^{-2\lambda T_2})}{N\lambda T_2}.$$

Since,

$$E(R)=1-e^{-\lambda T_1},$$

we have

$$\lambda T_2=-K\ \log_e(1-E(R)).$$

So,

$$\sigma_F^2 \approx V(\rho) = \frac{(1-e^{-\rho})-.5(1-e^{-2\rho})}{N\rho}$$

where,

$$\rho = -K \log_e(1-E(R)).$$

Now let,

$$F = E(F) + \sigma_F \xi_F \ .$$

Then, if NF is large, ξ_F is approximately normally distributed with mean zero and variance one.

Similarly, if RN is large and,

$$R = E(R) + \sigma_R \xi_R$$

where, (see Appendix B),

$$\sigma_R = \sqrt{E(R)(1-E(R))/N}$$

then, ξ_R is approximately standard normal as well.

Since F(R,K) has derivatives of all order in R, then, using the Taylor series expansion,

$$F(R,K) = F(E(R),K) + F'(E(R),K)\sigma_R \xi_R + O(\frac{1}{N})$$

where, F'(E(R),K) denotes the derivative of F(E(R),K) with respect to its first argument and O(1/N) denotes, unspecifically, terms of order 1/N.

Therefore,

$$F - F(R,K) = -F'(E(R),K)\sigma_R\xi_R + \sigma_F\xi_F - O(\frac{1}{N}).$$

Consequently,

$$F - F(R,K) = \sigma\eta - O(\frac{1}{N}) \approx \sigma\eta$$

where, η is a standard (mean zero, variance one) normal random variable and,

$$\sigma = \sqrt{\sigma_F^2 + (F'(E(R),K))^2\sigma_R^2}.$$

Therefore, when N is large,

$$Pr(F > F(R,K) + Z_\alpha\sigma) \approx \alpha.$$

Where Z_α is a standard normal deviate, i.e., if X is any standard normal random variable, then Z_α is a quantity such that $Pr(X>Z_\alpha)=\alpha$. Consequently, $Pr(\eta>Z_\alpha)=\alpha$ and since $\eta=(F-F(R,K))/\sigma$, we can say that $Pr((F-F(R,K))/\sigma>Z_\alpha)\approx\alpha$. From this argument, the expression given above follows immediately. (For values of Z_α, see Appendix C.)

If E(R) is known, σ and then consequently, F_U, an upper prediction limit for F, can be found. Since terms of the order 1/N in the calculation of the prediction limit are neglected and noting that σ is of the order of $1/N^{1/2}$, E(R) is also approximated in the expression for σ. Substitution of R for E(R) for the upper prediction limit would still be approximately correct for large N. However, the upper prediction limit would err on the low side since,

$$F_U \approx F(R,K) + Z_\alpha\sigma$$

and σ would become a function of R. Variations in R would underestimate σ, at times, leading to an increased incidence of F exceeding F_U. However, if

$$R_\beta = R + Z_\beta \sqrt{R(1-R)/N}$$

is substituted for E(R) in σ, a conservative estimate of F_U can be found. (For values of Z_β see Appendix C.) More precisely, R_β is substituted for E(R) in the expressions for σ_R and σ_F, and R is substituted for E(R) in F'(E(R),K) since the latter quantity decreases with E(R).

Note that R_β is itself in the form of a confidence limit for R. For small values of Z_β this is a mildly biased estimate of R. For larger values of Z_β it becomes a strongly biased estimate (almost surely an overestimate) or strongly conservative (safely pessimistic) estimate of R. Only simulation can determine the value of Z_β to use with each value of α. This is dealt with in the sections on simulation.

Note that F'(R,K) can be approximated adequately by

$$F'(R,K) \approx \frac{\Delta F}{\Delta R} = \frac{F(R+.5\delta,K) - F(R-.5\delta,K)}{\delta}, \quad where\ \delta = \sqrt{\frac{R}{N}}.$$

Therefore, the final expression for the upper prediction limit is

$$F_U = F(R,K) + Z_\alpha \sqrt{S_F^2 + (\Delta F/\Delta R)^2 S_R^2}$$

where,

$$S_F^2 = V(\rho)\ with\ \rho = -K \log_e(1-R_\beta)$$

and

$$S_R^2 = R_\beta(1 - R_\beta)/N.$$

As N approaches infinity, F_U becomes an exact upper α prediction limit. The error goes to zero faster than the offset, F_U-F(R,K).

The results of some simulation runs appear in Appendix D and suggest that this expression, when applied to calculate an upper 0.975 prediction limit, is nearly always conservative if Z_β is chosen to be 2.5. The formula and its performance are not very sensitive to this parameter. Further simulations suggest that Z_β values of 2.1, 2.2, and 2.3 for α=0.05, 0.03, and 0.02 respectively, work satisfactorily.

Note: R is the number of failed units divided by N. When no units fail, a slight modification (see Appendix C, Step 2) works better and is reported in "Experimental Statistics" by Mary Natrella, National Bureau of Standards Handbook 91, 1963. In such a case, instead of R=O, R=0.25/N is used. This modification, ascribed to Bartlett, who proposed it in a different context, improves performance in the simulation. Bartlett's modification also calls for a symmetric treatment of the symmetric case when all units fail. The full Bartlett modification should be used considering that failure of all units is unlikely to ever occur.

APPENDIX B. DERIVATION OF VARIANCE FORMULAS

Let D_I denote the number of units which fail between inspections. Then,

$$D_I = \sum_{i=1}^{N} X_i,$$

where X_i equals 1 if unit i fails between inspections and X_i equals 0 if unit i does not fail. Then,

$$E(X_i) = Pr(X_{i=1}) = 1 - e^{-\lambda T},$$

where T is the inspection interval and λ is the failure rate. Then D_I is binomially distributed Bin(n,p) with n=N and $p=1-e^{-\lambda T}$. We denote D_I/N by R. Then,

$$E(R) = E(\frac{D_I}{N}) = \frac{E(\sum_{i=1}^{N} X_i)}{N} = 1 - e^{-\lambda T}.$$

Now,

$$\sigma_R = \sqrt{E(R^2) - (E(R))^2} = \sqrt{E(R)(1 - E(R))/N}.$$

Since the X_i's are independent and,

$$NR = \sum_{i=1}^{N} X_i,$$

we have

$$VAR[NR] = N^2 \sigma_R^2 = \sum_{i=1}^{N} \sigma_{x_i}^2.$$

Since X_i is a Bernoulli or binary random variable we have

$$\sigma_{x_i}^2 = E(X_i)(1 - E(X_i))$$

and

$$\sum_{i=1}^{N} \sigma_{x_i}^2 = \sum_{i=1}^{N} E(X_i)(1 - E(X_i)) = N(1 - e^{-\lambda T})e^{-\lambda T}.$$

Then,

$$\sigma_R^2 = \frac{(1 - e^{-\lambda T})e^{-\lambda T}}{N} = \frac{E(R)(1 - E(R))}{N}.$$

Next, let D_t be the number of in-service units which are in a failed condition at some specific time. Then,

$$D_t = \sum_{i=1}^{N} X_i,$$

where $X_i = 1$ if unit i is failed at the given moment in time. Then,

$$E(X_i) = Pr(X_i = 1) = 1 - e^{-\lambda t_i},$$

where t_i is the amount of time since the last inspection of unit i. Since this is a uniform inspection, unit i will not have been inspected for an approximate interval of time of (i/N)T. Here, as a convenience (without loss of generality), we take the units as being numbered with the most recently inspected having the lowest numbers. Therefore,

$$E(D_t) = \sum_{i=1}^{N} E(X_i) = \sum_{i=1}^{N} (1 - e^{-\frac{i}{N}\lambda T}).$$

We approximate the sum by the corresponding integral,

$$E(\frac{D_t}{N})=\frac{1}{N}\sum_{i=1}^{N}(1-e^{-\frac{i}{N}\lambda T})\approx\frac{1}{T}\int_0^T(1-e^{-\lambda t})dt.$$

Actually, the integral is more accurate than the sum since the exact time that unit i was inspected is not exactly (i/N)T but is distributed between (i/N)T and ((i-1)/N)T, which the integral reflects.

Denote D_t/N by F. Then,

$$E(F)=E(\frac{D_t}{N})\approx\frac{1}{T}\int_0^T(1-e^{-\lambda T})dt=1-\frac{(1-e^{-\lambda T})}{\lambda T}.$$

Since

$$N\ E(F)=E(D_t)=\sum_{i=1}^{N}E(X_i),$$

then $\sigma_F^2=E(F^2)-E(F)^2$ is obtained in a similar way used to get σ_R^2, so that

$$\sigma_F^2=\frac{1}{N^2}\sum_{i=1}^{N}\sigma_{x_i}^2=\frac{1}{N^2}\sum_{i=1}^{N}E(X_i)(1-E(X_i)),$$

After substitution,

$$\sigma_F^2\approx\frac{1}{NT}\int_0^T(1-e^{-\lambda t})(1-(1-e^{-\lambda t}))dt=\frac{1}{NT}\int_0^T(e^{-\lambda t}-e^{-2\lambda t})dt$$

to yield

$$\sigma_F^2 \approx \frac{(1-e^{-\lambda t})-.5(1-e^{-2\lambda t})}{N\lambda T}$$

Note that if R refers to a given inspection interval of duration T_1 while F refers to a separate inspection interval of duration T_2 then we may solve the expression for R for λ. Since,

$$E(R)=1-e^{-\lambda T_1} \text{ we have } \lambda=\frac{-\log_e(1-E(R))}{T_1}.$$

This expression for λ, then, may be substituted into expressions for E(F) and σ_F and will be used in further developments.

APPENDIX C. DETERMINATION OF F_U AND T_2

In this section, the formula for F_U is applied in examples to demonstrate the calculation of an upper prediction limit and a new inspection interval. Both types of calculation are implemented in a BASIC program listing that is also included. Representative graphical evaluations are found in Section 7 to permit a rapid and approximate determination of F_U and T_2.

C.1 Procedure to Calculate F_U

The following procedure outlines the steps required to calculate the value of F_U, the upper prediction limit.

Step 1. Determine the values of the following quantities:

N	The population size
D	The number of units in the entire population which were found defective in one inspection cycle
T_1	The length of the inspection cycle in units of time,i.e., the original inspection cycle - when D was measured
T_2	The length of the new inspection cycle related to F

Step 2. Calculate R as follows:

If D>0 and D<N then, R=D/N (the usual case)
If D=0 then, R=.25/N (no defects, an unusual case)
If D=N then, R=l-.25/N (should never happen, included for completeness)

Step 3. Calculate K:

$$K = T_2 / T_1$$

Step 4. Calculate the conditional expected value of F:

$$F(R,K) = 1 + \frac{1-(1-R)^K}{K \log_e(1-R)}$$

Step 5. Calculate value of Z_α:

If α=.05, then Z_α=1.644854

If α=.03, then Z_α=1.880794

If α=.02, then Z_α=2.053749

Z_α is determined by the normal distribution in the following familiar way: If X is a standard normal random variable with mean=0 and variance=l, then $\Pr(X>Z_\alpha)=\alpha$ (see Figure C-1).

Step 6. Determine value of Z_β by α:

If α=0.05, then Z_β=2.1

If α=0.03, then Z_β=2.2

If α=0.02, then Z_β=2.3

These values are determined by simulation and are not nearly as critical as the values of Z_α.

Step 7. Calculate the compensated value of R for use in parts of the formula:

$$R_\beta = R + Z_\beta \sqrt{\frac{R(1-R)}{N}}.$$

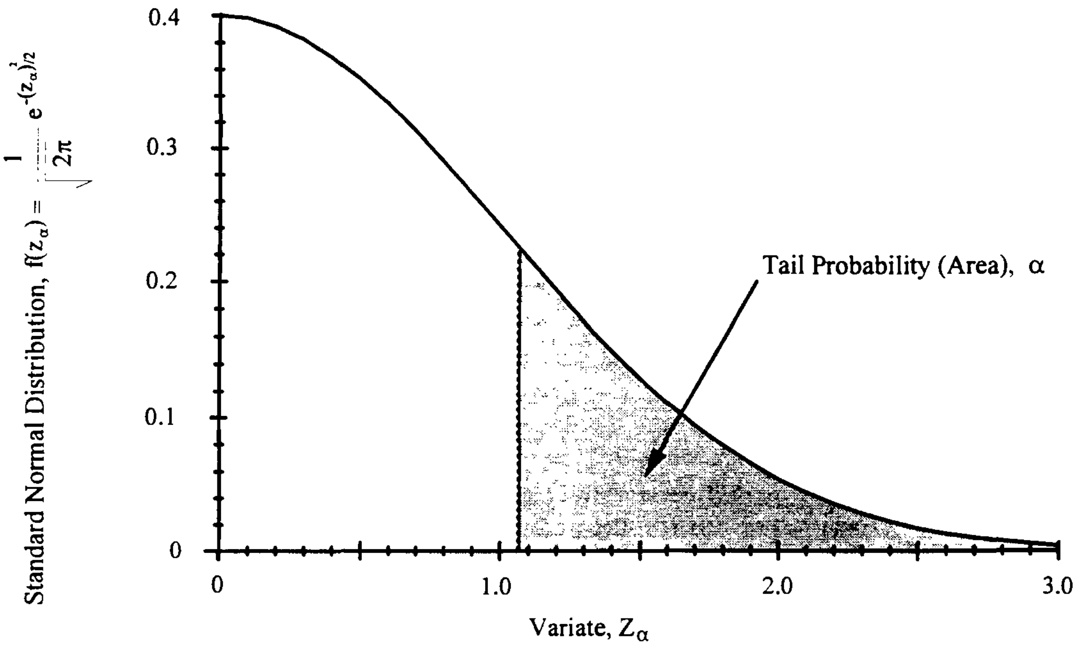


Figure C-1a. Standard Normal Distribution

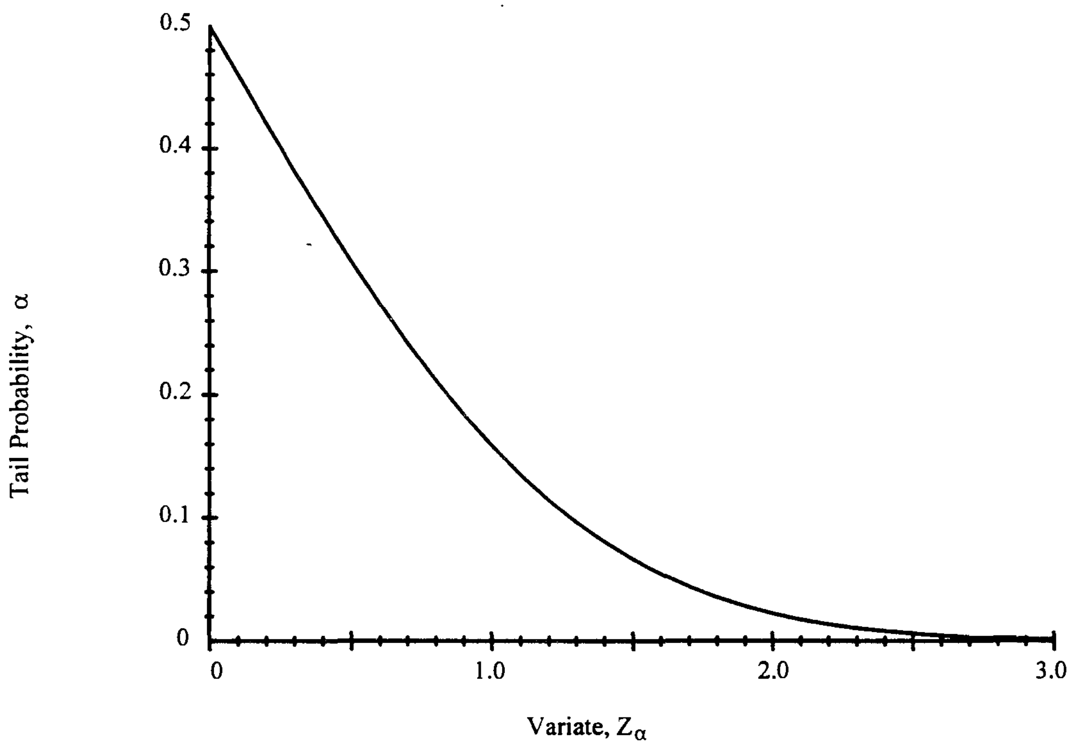


Figure C-1b. Tail Probability

FIGURE C-1. NORMAL PROBABILITY DISTRIBUTION

Step 8. Calculate an estimated derivative of F(R,K):

$$\frac{\Delta F}{\Delta R}=\frac{F(R+.5\delta,K)-F(R-.5\delta,K)}{\delta}, \textit{ where } \delta=\sqrt{\frac{R}{N}}.$$

Step 9. Calculate the point estimate of the variance in F:

$$S_F^2=V(\rho), \textit{ where } \rho=-K\log_e(1-R_\beta),$$

and

$$V(\rho)=\frac{(1-e^{-\rho})-.5(1-e^{-2\rho})}{N\rho}.$$

Step 10. Calculate the point estimate of the variance in R:

$$S_R^2=\frac{R_\beta(1-R_\beta)}{N}.$$

Step 11. The complete formula for F_U is:

$$F_U=F(R,K)+Z_\alpha\sqrt{S_F^2+(\frac{\Delta F}{\Delta R})^2 S_R^2}.$$

Note: If R is very small and N is sufficiently large, then the above formula is well approximated by:

$$F_U\sim\frac{KR}{2}+Z_\alpha\sqrt{\frac{KR}{2N}+(\frac{K}{2})^2\frac{R}{N}}.$$

This latter estimate should not be used for calculation because it can be much less accurate than the formula given in Step 11 and is intended only to be a rough and qualitative indicator of how the more accurate formula behaves as a function of its arguments when R is small.

C.2 Procedure to Calculate T_2

The following procedure outlines the steps required to calculate the value of T_2, a new inspection interval given D, N, F_U, and α.

The problem is how to determine T_2 when a desired value for F_U is given. This value will be referred to as F_T, the target value. The formula, given in Step 11, is solved for K. A value of T_2 is found with the equation, $T_2=KT_1$. To solve for K we proceed iteratively as follows:

Step 1. Choose an initial value for K, such as K=1.

Step 2. Calculate F_U using the formula of Step 11, above, and the assumed value for K or the current best value.

Step 3. If F_T/F_U is very close to 1 so that $|(F_T-F_U)|/F_U \leq 0.001$, then the calculation is complete and the current value of K is the solution.

Step 4. Otherwise, adjust K by the following formula: New K = Old $K^*(F_T/F_U)$, then go back to Step 2. Iterate through Steps 2, 3 and 4 as many times as necessary to bring the ratio, F_T/F_U, satisfactorily close to 1. The resulting value of K is the ratio of T_2/T_1, where T_1 is the old inspection interval and T_2 is the length of the inspection interval which results in the specified (or target) value, F_T of F_U.

C.3 A BASIC Program to Calculate F_U and K

This section is devoted to a description of how to use a BASIC program to calculate F_U and K. F_U is calculated as a function of N, R, K, and α. K is calculated as a function of F_T, N, R, and α. The program is listed in Figure C-2. It is written in Quick Basic and is readily converted to any other dialect of BASIC. Since it requires a simple numerical calculation with only minimal keyboard input and output, it is also easily converted to FORTRAN, C, or Pascal. (The program prompts twice for keyboard entry of numerical inputs.)

The following numerical examples illustrate the procedures to calculate the values of the upper prediction limit and the new inspection interval. Examples 1 and 2 show how respective values for F_U and K are calculated. Example 3 shows the inverse relationship between the calculations of F_U and K.

Example 1. Given the following values: N=1000, R=0.05, K=1.5, α=0.03, determine F_U based on the procedure to calculate the upper prediction limit.

First, determine Z_α and Z_β with α by Steps 5 and 6 of the procedure in Appendix C.l. For the value of α=0.03, then Z_α=1.880794 and z_β=2.2.

Now, run the Quick Basic Program for calculating F_U. When asked if the user wants to calculate F_U, or K, enter 1 to request the former (as directed by the prompt). After the second prompt asks for R, N, K, Z_α, and Z_β, enter their respective values: 0.05, 1000, 1.5, 1.880794, and 2.2. The result is F_U=0.05432.

Example 2. Determine the inspection interval ratio using R=0.1, N=300, F_T=0.05, Z_α=1.644854, and Z_β=2.1. (Z_α and Z_β are evaluated for α=0.05. See steps 5 and 6 of Appendix C.l.)

Run the Quick Basic Program to calculate K by entering 2 at the first prompt. At the second prompt, enter the following values: 0.1, 300, and 0.05, respectively, for R, N, F_T, as well as the values of Z_α=1.644854 and Z_β=2.1. The result is K=0.56256.

FIGURE C-2. QUICK BASIC PROGRAM FOR CALCULATING K OR F_U

```
DECLARE SUB FFNL (XXP!, KXP!, FXP!)
R=0.1: N=100: T2=1: ZA=1.645: ZB=2.5: FT=0.05: RN=1.2: EP=0.00001
PRINT
ST:
PRINT "ENTER 1 IF YOU WANT TO CALCULATE FU; ENTER 2 IF YOU WANT TO CALCULATE K";
INPUT IC
IF IC=2 THEN
PRINT "INPUT R, N, FT, ZA, ZB"
INPUT R, N, FT, ZA, ZB
KR=1
GOSUB  SOLV
PRINT  "K="; KR
END
ELSE
IF IC  <> 1 THEN GOTO ST
PRINT  "INPUT R, N, K, ZA, ZB"
INPUT  R, N, KR, ZA, ZB
GOSUB FUF: PRINT "FU="; FU
END IF
END

SOLV:
ITT:
GOSUB FUF
RF=FT/FU: KR=KR*RF
IF RF <1-EP OR RF>1+EP THEN GOTO ITT
RETURN
END

UT: PRINT "X"; ITS; KR; FUU; JN; N
PRINT #1, N, KR
CLOSE
END

FCC:
FU=1-((1-RU)^KU-1)/(KU*LOG(1-RU))
RETURN
```

FIGURE C-2. QUICK BASIC PROGRAM FOR CALCULATING K OR F_U (Cont.)

```
FUF:
DP = SQR (R*(1-R)/N)
RB=R+ZB*DP
CALL FFNL (R, KR, FF)
RLT=-KR*LOG(1-RB)
VF=((1-EXP(-RLT))-O.5*(1-EXP(-2*RLT)))/RLT
DP=SQR(R*(1-R)/N)
PU=R+DP*0.5
PL=R-DP*0.5
CALL FFNL(PU, KR, FUU)
CALL FFNL(PL, KR, FUL)
DF=(FUU-FUL)/DP
FU=FF+ZA*SQR((VF+RB*(1-RB)*DF*DF)/N)
RETURN
SUB FFNL (XXP, KXP, FXP)

U=XXP: KR=KXP
FXP=1-((1-U)^KR-1)/(KR*LOG(1-U))
END SUB
```

Example 3. To demonstrate the relationship between the calculations for F_U and K, data from example 2, used to calculate K, will be used to find the value of F_U.

Respond with 1 to the first prompt to request calculation of F_U. Enter the specified values of R, N, Z_α, and Z_β, along with the computed value of K=0.56256, all found in example 2. The output value of F_U is the same as the input value of 0.05 for F_T in example 2 and shows that the two types of calculation performed by this program are the inverse of each other.

APPENDIX D. SIMULATION TESTING OF THE FORMULA FOR F_U

This appendix describes a simulation developed to study the performance of the formula for F_U. Although the formula is compensated for the effect of finite sample size, its form was derived under ideal conditions assuming that the sample size is sufficiently large not to require compensation. It is common practice to derive an expression that is proved to be accurate asymptotically, e.g., as the sample size goes to infinity, and to use that expression with the expectation that it will be approximate, yet sufficiently accurate, in non-asymptotic cases when the sample size is finite. It is necessary to know for what ranges of parameters, especially, for the population size, N, that the formula remains accurate. The simulation must be faithful and based on a model with the exact characteristics, e.g., constant failure rate, etc., of the conceptual model.

The simulation is constructed according to the following procedure:

First, prescribe values for the four parameters: N, the size of the population being simulated; E(R), the expected value of R; T_1, the length of the inspection interval when R is observed; and T_2, the length of the inspection interval when F is to occur.

Next, determine values of R and F by different independent simulations. R is determined by

$$R=\frac{\sum_{i=1}^{N} X_i}{N}$$

where each X_i is an independent binary random variable indicating that unit i failed ($X_i=1$) between inspections or did not fail ($X_i=0$). Thus, R is simulated by taking the sum of N independent binary random variables and dividing the sum by N. Each X_i is simulated by adding a 1 with probability E(R) and a zero with probability 1-E(R), i.e., $Pr(X_i=1)=E(R)$. Recall (from Appendix B) that E(R) and λ are related by

$$E(R)=1-e^{-\lambda T_1}.$$

F is determined in a separate simulation in a similar manner with some important differences. F is also of the form,

$$F=\frac{\sum_{i=1}^{N} X_i}{N}$$

where each X_i is an independent binary random variable; however, each X_i does not have the same probability of taking the value 1. The units are numbered in the order in which they are inspected. (i=1 indexes the first unit inspected; i=N indexes the last unit inspected.) Then,

$$Pr(X_i=1)=1-e^{-\lambda \frac{i-d}{N} T_2}$$

where d is a number between 0 and 1. The value of d has very little to do with the outcome since, on the average, i is of the order of N/2, but for completeness, a single random value is assigned to d each time F is simulated. Now, both R and F have been generated using standard statistical programming techniques.

Next, the value of the upper prediction limit is determined by the formula for F_U (see Appendix A). For given values of N, E(R), T_1, and T_2, many independent pairs of R and F are generated, each in the manner described above. The number of independent complete simulations will be denoted by NSIM.

Based on the simulations, two statistics, α_s, and q, are calculated.

For very large values of NSIM, the observed fraction, α_s, the fraction of cases where F exceeds F_U, approaches the true value of α for the given parameters.

The qth quantile from the top is the largest value of (F-F_U) which determines the fraction q. In the cases presented, q is the intended fraction of cases where F exceeds F_U and is the intended value of α. The qth quantile value is, then, the amount by which F_U would need to be increased for it to actually be an α upper prediction limit on F.

The following three steps are used to perform the simulation: (1) choose 3 values for N, 3 values for T_2, and 3 values for E(R) with T_1 fixed at 1 without loss of generality; (2) at all 27 combinations of values of these parameters, simulate NSIM pairs R, F with NSIM chosen to be greater than 10000, preferably, 100000; and (3) calculate the two statistics, α, and q, to summarize the run at each combination of parameters.

Table D-1 presents the results of some simulations. For each case represented in this table, α is 0.025. Z_β is taken to be 2.5. (A somewhat smaller value is examined for this case in Appendix E.) NSIM is 80000 which means that each line in the table reports on 80000 separate simulations.

Note that the last three of seven columns in Table D-1 indicate the three input parameters. Columns 5, 6, and 7 list T_2, E(R), and N, respectively, for each of the 27 cases. The results are listed in the first four columns.

Column 1 contains the most important result, the value α_s, i.e., the observed fraction of NSIM cases in which F exceeds F_U. The goal is to obtain a conservative upper α prediction limit α=0.025. The value of the fraction should be less than 0.025 in most instances. In this run, there was only one exception.

Column 2 gives the 0.025 quantile of F_U-F and is positive except in cases where the first column exceeds 0.025. A positive value indicates the amount by which F_U exceeds the 0.975 upper prediction limit of F. Since the absolute numerical value of the 0.025 quantile is always small, it appears that the formula for F_U-F, although conservative, is not causing F_U to be too large by an appreciable amount for the cases considered. Negative values in column 2 indicate the cases where F_U falls short of being conservative and (in their absolute value) the

TABLE D-1. SIMULATION RESULTS

$\alpha = .025$ $Z_\alpha = 1.96$ $Z_\beta = 2.5$ NSIM = 80,000

Observed Fraction Exceeding Limit	.025 Quantile of $(F_U - F)$	Average Value of $(F_U - F)$	Average Value of F_U	T_2	E(R)	N
0.02224	0.003437	0.10537	0.17895	3.00	0.0500	100
0.01616	0.009582	0.07361	0.09611	3.00	0.0150	100
0.02164	0.001996	0.12733	0.56846	3.00	0.3500	100
0.01912	0.002435	0.05595	0.12917	3.00	0.0500	300
0.02375	0.000023	0.03708	0.05945	3.00	0.0150	300
0.02195	0.001587	0.07637	0.51430	3.00	0.3500	300
0.01883	0.002433	0.04186	0.11508	3.00	0.0500	500
0.02006	0.001189	0.02736	0.04968	3.00	0.0150	500
0.02427	0.000818	0.05856	0.49742	3.00	0.3500	500
0.02634	-0.000153	0.05221	0.07728	1.00	0.0500	100
0.01096	0.001930	0.03412	0.04165	1.00	0.0150	100
0.02199	0.001791	0.09904	0.28706	1.00	0.3500	100
0.01818	0.001651	0.02721	0.05234	1.00	0.0500	300
0.02488	0.000146	0.01706	0.02456	1.00	0.0150	300
0.02378	0.000424	0.05544	0.24345	1.00	0.3500	300
0.01904	0.001103	0.02022	0.04542	1.00	0.0500	500
0.02034	0.000147	0.01254	0.02007	1.00	0.0150	500
0.02395	0.000471	0.04252	0.23038	1.00	0.3500	500
0.01775	0.000660	0.03460	0.04731	0.50	0.0500	100
0.01416	0.002914	0.02229	0.02604	0.50	0.0150	100
0.02338	0.001825	0.07389	0.17500	0.50	0.3500	100
0.02159	0.000738	0.01791	0.03063	0.50	0.0500	300
0.01812	0.000255	0.01115	0.01491	0.50	0.0150	300
0.02284	0.000840	0.04111	0.14141	0.50	0.3500	300
0.01980	0.000578	0.01329	0.02602	0.50	0.0500	500
0.02107	0.000500	0.00815	0.01193	0.50	0.0150	500
0.02386	0.000393	0.03114	0.13166	0.50	0.3500	500

amount by which F_U would need to be increased to make it conservative. There is only one such case in this run and the amount (1.5 X 10-4) is negligible.

Columns 3 and 4 give the average values of (F_U-F) and of F_U, respectively. The average value of (F_U-F) is especially interesting since it shows by how much the observed value of F is increased, on average, to produce the prediction limit and is an indication of the necessary cost of conservatism. We have referred to this as the confidence margin. This indicates the amount by which the upper prediction limit exceeds the mean average value.

Table D-2 is similar to Table D-1. The values of Z_α and Z_β are the same, 1.96 and 2.5, respectively. Other parameters are somewhat different. The values of N are 200, 600, and 1800. The values of E(R) are 0.01, 0.03, and 0.09. The values of T_2 are 0.5, 1.0, and 2.0. The number of repetitions of each case is 200000. Table D-2 shows that the formula yields conservative values for F_U for all values of the three variable parameters considered. In no case did the value in column 1 exceed 0.025. Comparison of column 2 with column 3, as in the case of Table D-1, can show that if the formula is modified into an exact upper prediction limit the modification will be small. (The extent of the modification to column 2 will be small compared to the average amount by which F_U already exceeds F in column 3.)

TABLE D-2. SIMULATION RESULTS

$\alpha = .025 \quad Z_\alpha = 1.96 \quad Z_\beta = 2.5 \quad NSIM = 200,000$

Observed Fraction Exceeding Limit	.025 Quantile of (F_U - F)	Average Value of (F_U - F)	Average Value of F_U	T_2	E(R)	N
0.02008	0.003136	0.03056	0.04060	2.00	0.0100	200
0.01774	0.001923	0.04527	0.07503	2.00	0.0300	200
0.01835	0.003807	0.06584	0.15403	2.00	0.0900	200
0.01922	0.000722	0.01552	0.02552	2.00	0.0100	600
0.01974	0.001394	0.02369	0.05348	2.00	0.0300	600
0.02043	0.001417	0.03541	0.12420	2.00	0.0900	600
0.01880	0.000481	0.00811	0.01805	2.00	0.0100	1800
0.02018	0.000496	0.01264	0.04251	2.00	0.0300	1800
0.02159	0.000583	0.01961	0.10822	2.00	0.0900	1800
0.01646	0.000992	0.01899	0.02398	1.00	0.0100	200
0.02193	0.000202	0.02829	0.04345	1.00	0.0300	200
0.02019	0.002239	0.04296	0.08872	1.00	0.0900	200
0.02214	0.000113	0.00961	0.01461	1.00	0.0100	600
0.01910	0.000948	0.01480	0.02988	1.00	0.0300	600
0.01997	0.001161	0.02314	0.06883	1.00	0.0900	600
0.02042	0.000244	0.00497	0.00999	1.00	0.0100	1800
0.02106	0.000298	0.00789	0.02298	1.00	0.0300	1800
0.02231	0.000316	0.01276	0.05844	1.00	0.0900	1800
0.01643	0.001481	0.01230	0.01482	0.50	0.0100	200
0.02000	0.000358	0.01862	0.02621	0.50	0.0300	200
0.02096	0.001180	0.02885	0.05206	0.50	0.0900	200
0.01953	0.000126	0.00622	0.00874	0.50	0.0100	600
0.02050	0.000641	0.00969	0.01727	0.50	0.0300	600
0.02058	0.000692	0.01541	0.03863	0.50	0.0900	600
0.02087	0.000141	0.00322	0.00574	0.50	0.0100	1800
0.02255	0.000154	0.00516	0.01273	0.50	0.0300	1800
0.02266	0.000195	0.00843	0.03166	0.50	0.0900	1800

APPENDIX E. RESULTS OF SEVERAL SIMULATION TEST RUNS

This appendix reports on several simulation runs of the type introduced in Appendix D. Each run of the simulation program is represented by a separate table. Each table is headed by a statement of the value of α and the two quantities, Z_α and Z_β, which depend on α and apply to all the simulation runs represented by the table. The values of Z_α and Z_β are as determined in Appendix C.

Each row represents 80,000 complete simulations under the conditions stated in the last three entries in the row for T_2, E(R), and N, respectively, similar to the entries in Table D-1 in Appendix D. The first four columns are also as described in Appendix D, with the exception of different values for α. The first column contains the actual proportion of the 80,000 runs in which F exceeded F_U and is the quantity which should be less than α. Column 5 contains an entry, not given in Appendix D, for the average value of F.

Note that in the tables of this appendix, the third, fourth, and fifth columns contain the average values of F_U, $(F_U\text{-}F)$, and F, respectively, and that columns 4 and 5 add up to the corresponding value in column 3. Also, it may be noted that the values of Z_β, here, are generally smaller than those in Appendix D and, so, somewhat more of the cases show a proportion of exceedances (column 1) in excess of α. However, the amount of change in F_U needed to produce an exact α prediction limit (column 2) is small compared to the average value of F (column 5) in nearly every case. Therefore, the somewhat less conservative values are recommended for use in this appendix and in the section describing the calculations. A slightly larger value of Z_β, however, could be used and should result in a more conservative upper prediction limit which would differ little from that examined here.

TABLE E-1. SIMULATION RESULTS

α = .02 Z_α = 2.05375 Z_β = 2.3 NSIM = 80,000

1 Observed Fraction where F exceeds F_U	2 .02 Quantile of (F_U - F)	3 Average value of F_U	4 Average value of (F_U - F)	5 Average value of F	6 T_2	7 E(R)	8 N
0.00655	0.006541	0.06010	0.05018	0.00993	2.0	0.010	100
0.01436	0.006541	0.07089	0.05585	0.01504	2.0	0.015	100
0.02045	-0.003459	0.08102	0.06091	0.02011	2.0	0.020	100
0.01509	0.004432	0.04764	0.03764	0.01001	2.0	0.010	150
0.02059	-0.002235	0.05798	0.04305	0.01493	2.0	0.015	150
0.02183	-0.002235	0.06750	0.04758	0.01992	2.0	0.020	150
0.02022	-0.001646	0.04123	0.03114	0.01009	2.0	0.010	200
0.02405	-0.001646	0.05097	0.03601	0.01496	2.0	0.015	200
0.02052	-0.001646	0.05998	0.04003	0.01995	2.0	0.020	200
0.00594	0.002193	0.03608	0.03109	0.00499	1.0	0.010	100
0.01091	0.002193	0.04237	0.03486	0.00750	1.0	0.015	100
0.01570	0.002193	0.04834	0.03829	0.01006	1.0	0.020	100
0.01072	0.001483	0.02838	0.02343	0.00495	1.0	0.010	150
0.01846	0.001483	0.03421	0.02668	0.00754	1.0	0.015	150
0.01999	0.000189	0.03970	0.02963	0.01007	1.0	0.020	150
0.01619	0.001121	0.02435	0.01936	0.00499	1.0	0.010	200
0.01965	0.000202	0.02991	0.02243	0.00749	1.0	0.015	200
0.01957	0.000202	0.03500	0.02507	0.00993	1.0	0.020	200
0.01030	0.003298	0.02269	0.02020	0.00249	0.5	0.010	100
0.01464	0.003298	0.02651	0.02274	0.00377	0.5	0.015	100
0.01655	0.003298	0.03009	0.02507	0.00502	0.5	0.020	100
0.00261	0.002197	0.06300	0.06049	0.00251	0.5	0.010	150
0.00575	0.002197	0.04226	0.03850	0.00377	0.5	0.015	150
0.00988	0.002197	0.03441	0.02936	0.00505	0.5	0.020	150
0.00495	0.001639	0.06460	0.06206	0.00254	0.5	0.010	200
0.00945	0.001639	0.03668	0.03288	0.00380	0.5	0.015	200
0.01380	0.000632	0.02777	0.02272	0.00504	0.5	0.020	200

TABLE E-2. SIMULATION RESULTS

α = .03 Z_α = 1.8808 Z_β = 2.2 NSIM = 80,000

1	2	3	4	5	6	7	8
Observed Fraction where F exceeds F_U	.02 Quantile of $(F_U - F)$	Average value of F_U	Average value of $(F_U - F)$	Average value of F	T_2	E(R)	N
0.00684	0.013124	0.05533	0.04540	0.00993	2.0	0.010	100
0.01529	0.003124	0.06551	0.05051	0.01500	2.0	0.015	100
0.02362	0.003124	0.07513	0.05504	0.02008	2.0	0.020	100
0.01588	0.002143	0.04409	0.03404	0.01005	2.0	0.010	150
0.02659	0.002143	0.05376	0.03881	0.01495	2.0	0.015	150
0.03060	-0.000963	0.06296	0.04306	0.01990	2.0	0.020	150
0.02229	0.001632	0.03825	0.02833	0.00992	2.0	0.010	200
0.03142	-0.000522	0.04753	0.03258	0.01495	2.0	0.015	200
0.03271	-0.000522	0.05613	0.03625	0.01988	2.0	0.020	200
0.00579	0.000085	0.03308	0.02807	0.00501	1.0	0.010	100
0.01106	0.000085	0.03899	0.03151	0.00748	1.0	0.015	100
0.01696	0.000085	0.04464	0.03456	0.01008	1.0	0.020	100
0.01161	0.000077	0.02613	0.02111	0.00502	1.0	0.010	150
0.01920	0.000077	0.03167	0.02424	0.00743	1.0	0.015	150
0.02378	0.000077	0.03690	0.02685	0.01005	1.0	0.020	150
0.01695	0.000067	0.02250	0.01744	0.00506	1.0	0.010	200
0.02375	0.000067	0.02780	0.02028	0.00752	1.0	0.015	200
0.02743	0.000067	0.03266	0.02256	0.01010	1.0	0.020	200
0.01003	0.002826	0.02072	0.01826	0.00247	0.5	0.010	100
0.01608	0.001258	0.02427	0.02052	0.00375	0.5	0.015	100
0.02050	0.001258	0.02765	0.02256	0.00509	0.5	0.020	100
0.00391	0.005802	0.05706	0.05456	0.00250	0.5	0.010	150
0.01030	0.000832	0.03835	0.03458	0.00377	0.5	0.015	150
0.01657	0.000832	0.03154	0.02652	0.00503	0.5	0.020	150
0.00775	0.002451	0.05810	0.05561	0.00248	0.5	0.010	200
0.01634	0.000618	0.03284	0.02910	0.00374	0.5	0.015	200
0.02255	0.000618	0.02595	0.02091	0.00504	0.5	0.020	200

TABLE E-3. SIMULATION RESULTS

$\alpha = .05$ $Z_\alpha = 1.64485$ $Z_\beta = 2.1$ NSIM = 80,000

1 Observed Fraction where F exceeds F_U	2 .02 Quantile of $(F_U - F)$	3 Average value of F_U	4 Average value of $(F_U - F)$	5 Average value of F	6 T_2	7 E(R)	8 N
0.02727	0.008788	0.04911	0.03923	0.00987	2.0	0.010	100
0.04287	0.008788	0.05857	0.04368	0.01489	2.0	0.015	100
0.04808	0.000008	0.06760	0.04761	0.02000	2.0	0.020	100
0.04364	0.005911	0.03940	0.02934	0.01006	2.0	0.010	150
0.04784	0.000213	0.04852	0.03343	0.01509	2.0	0.015	150
0.04134	0.000213	0.05713	0.03714	0.01999	2.0	0.020	150
0.04749	0.000247	0.03436	0.02434	0.01002	2.0	0.010	200
0.04449	0.000247	0.04315	0.02816	0.01499	2.0	0.015	200
0.04025	0.000247	0.05127	0.03129	0.01998	2.0	0.020	200
0.03726	0.007419	0.02921	0.02422	0.00499	1.0	0.010	100
0.05272	-0.000186	0.03467	0.02710	0.00758	1.0	0.015	100
0.05733	-0.000186	0.03987	0.02990	0.00997	1.0	0.020	100
0.05150	-0.000065	0.02325	0.01827	0.00498	1.0	0.010	150
0.05916	-0.000065	0.02839	0.02090	0.00749	1.0	0.015	150
0.05541	-0.000065	0.03327	0.02329	0.00998	1.0	0.020	150
0.05889	-0.000026	0.02012	0.01516	0.00496	1.0	0.010	200
0.05476	-0.000026	0.02501	0.01753	0.00749	1.0	0.015	200
0.04480	0.000657	0.02957	0.01964	0.00993	1.0	0.020	200
0.01949	0.001100	0.01822	0.01576	0.00246	0.5	0.010	100
0.03221	0.001100	0.02149	0.01771	0.00377	0.5	0.015	100
0.04127	0.001100	0.02456	0.01957	0.00499	0.5	0.020	100
0.02030	0.005758	0.04949	0.04698	0.00252	0.5	0.010	150
0.03135	0.003631	0.03362	0.02987	0.00375	0.5	0.015	150
0.03810	0.003631	0.02776	0.02272	0.00504	0.5	0.020	150
0.02980	0.002732	0.05046	0.04793	0.00252	0.5	0.010	200
0.03721	0.002732	0.02934	0.02555	0.00379	0.5	0.015	200
0.03847	0.000583	0.02275	0.01772	0.00503	0.5	0.020	200

TABLE E-4. SIMULATION RESULTS

$\alpha = .02$ $Z_\alpha = 2.05375$ $Z_\beta = 2.2$ NSIM = 80,000

1	2	3	4	5	6	7	8
Observed Fraction where F exceeds F_U	.02 Quantile of $(F_U - F)$	Average value of F_U	Average value of $(F - F_U)$	Average value of F	T_2	E(R)	N
0.00686	0.005940	0.05936	0.04934	0.01001	2.0	0.010	100
0.01370	0.005940	0.07008	0.05524	0.01483	2.0	0.015	100
0.01896	0.000723	0.08022	0.06042	0.01980	2.0	0.020	100
0.01436	0.004026	0.04722	0.03726	0.00996	2.0	0.010	150
0.02211	-0.002641	0.05730	0.04239	0.01491	2.0	0.015	150
0.02293	-0.002641	0.06682	0.04680	0.02002	2.0	0.020	150
0.01973	0.000678	0.04084	0.03083	0.01001	2.0	0.010	200
0.02324	-0.001953	0.05059	0.03561	0.01497	2.0	0.015	200
0.02099	-0.001953	0.05952	0.03956	0.01995	2.0	0.020	200
0.00593	0.001817	0.03560	0.03058	0.00503	1.0	0.010	100
0.01137	0.001817	0.04190	0.03442	0.00748	1.0	0.015	100
0.01680	0.001817	0.04776	0.03772	0.01004	1.0	0.020	100
0.01149	0.001233	0.02810	0.02308	0.00502	1.0	0.010	150
0.01875	0.001233	0.03405	0.02650	0.00755	1.0	0.015	150
0.02374	-0.002279	0.03943	0.02932	0.01012	1.0	0.020	150
0.01681	0.000935	0.02409	0.01910	0.00499	1.0	0.010	200
0.02246	-0.000066	0.02961	0.02208	0.00754	1.0	0.015	200
0.02436	-0.000066	0.03472	0.02468	0.01004	1.0	0.020	200
0.00975	0.003950	0.02237	0.01991	0.00247	0.5	0.010	100
0.01435	0.002983	0.02617	0.02245	0.00372	0.5	0.015	100
0.01709	0.000852	0.02971	0.02470	0.00501	0.5	0.020	100
0.00264	0.001982	0.06206	0.05954	0.00252	0.5	0.010	150
0.00587	0.001982	0.04206	0.03832	0.00374	0.5	0.015	150
0.00851	0.001982	0.03376	0.02875	0.00501	0.5	0.020	150
0.00543	0.001480	0.06324	0.06069	0.00255	0.5	0.010	200
0.00922	0.001480	0.03581	0.03201	0.00380	0.5	0.015	200
0.01339	0.000454	0.02745	0.02241	0.00504	0.5	0.020	200

TABLE E-5. SIMULATION RESULTS

$\alpha = .03$ $Z_\alpha = 1.8808$ $Z_\beta = 2.2$ NSIM = 80,000

1	2	3	4	5	6	7	8
Observed Fraction where F exceeds F_U	.02 Quantile of $(F_U - F)$	Average value of F_U	Average value of $(F_U - F)$	Average value of F	T_2	E(R)	N
0.00666	0.013124	0.05528	0.04528	0.01000	2.0	0.010	100
0.01592	0.003124	0.06546	0.05052	0.01493	2.0	0.015	100
0.02342	0.003124	0.07505	0.05509	0.01996	2.0	0.020	100
0.01474	0.002143	0.04417	0.03426	0.00992	2.0	0.010	150
0.02620	0.002143	0.05380	0.03896	0.01484	2.0	0.015	150
0.03105	-0.000963	0.06288	0.04307	0.01982	2.0	0.020	150
0.02307	0.001632	0.03824	0.02825	0.00999	2.0	0.010	200
0.03139	-0.000522	0.04755	0.03267	0.01488	2.0	0.015	200
0.03216	-0.000522	0.05614	0.03636	0.01978	2.0	0.020	200
0.00555	0.000085	0.03304	0.02806	0.00498	1.0	0.010	100
0.01156	0.000085	0.03899	0.03145	0.00754	1.0	0.015	100
0.01634	0.000085	0.04458	0.03455	0.01003	1.0	0.020	100
0.01101	0.000077	0.02617	0.02120	0.00498	1.0	0.010	150
0.01939	0.000077	0.03182	0.02429	0.00754	1.0	0.015	150
0.02331	0.000077	0.03699	0.02697	0.01002	1.0	0.020	150
0.01706	0.000067	0.02249	0.01744	0.00505	1.0	0.010	200
0.02391	0.000067	0.02781	0.02030	0.00751	1.0	0.015	200
0.02747	0.000067	0.03266	0.02258	0.01009	1.0	0.020	200
0.01119	0.002826	0.02072	0.01821	0.00251	0.5	0.010	100
0.01535	0.001258	0.02427	0.02054	0.00373	0.5	0.015	100
0.02060	0.001258	0.02765	0.02258	0.00506	0.5	0.020	100
0.00451	0.005802	0.05718	0.05467	0.00251	0.5	0.010	150
0.01021	0.000832	0.03846	0.03472	0.00374	0.5	0.015	150
0.01646	0.000832	0.03153	0.02649	0.00505	0.5	0.020	150
0.00774	0.002451	0.05814	0.05563	0.00251	0.5	0.010	200
0.01650	0.000618	0.03281	0.02907	0.00374	0.5	0.015	200
0.02217	0.000618	0.02586	0.02083	0.00503	0.5	0.020	200

TABLE E-6. SIMULATION RESULTS

α = .05 Z_α = 1.64485 Z_β = 2.2 NSIM = 80,000

1 Observed Fraction where F exceeds F_U	2 .02 Quantile of (F_U - F)	3 Average value of F_U	4 Average value of (F_U - F)	5 Average value of F	6 T_2	7 E(R)	8 N
0.02730	0.009282	0.04966	0.03977	0.00989	2.0	0.010	100
0.04091	0.009282	0.05919	0.04435	0.01484	2.0	0.015	100
0.04797	0.000621	0.06817	0.04816	0.02001	2.0	0.020	100
0.04290	0.006242	0.03995	0.02995	0.01000	2.0	0.010	150
0.04865	0.000631	0.04893	0.03392	0.01501	2.0	0.015	150
0.04302	0.000631	0.05753	0.03770	0.01983	2.0	0.020	150
0.04747	0.000565	0.03469	0.02471	0.00999	2.0	0.010	200
0.04465	0.000565	0.04347	0.02847	0.01500	2.0	0.015	200
0.03950	0.000565	0.05162	0.03166	0.01995	2.0	0.020	200
0.03296	0.007722	0.02957	0.02456	0.00500	1.0	0.010	100
0.03955	0.000202	0.03504	0.02761	0.00743	1.0	0.015	100
0.04025	0.000202	0.04031	0.03033	0.00998	1.0	0.020	100
0.04103	0.000197	0.02351	0.01849	0.00503	1.0	0.010	150
0.04024	0.000197	0.02878	0.02123	0.00754	1.0	0.015	150
0.03520	0.000197	0.03363	0.02356	0.01007	1.0	0.020	150
0.04181	0.000171	0.02032	0.01530	0.00502	1.0	0.010	200
0.03620	0.000171	0.02524	0.01774	0.00750	1.0	0.015	200
0.03191	0.000886	0.02982	0.01988	0.00994	1.0	0.020	200
0.02008	0.001293	0.01847	0.01597	0.00250	0.5	0.010	100
0.03171	0.001293	0.02173	0.01800	0.00373	0.5	0.015	100
0.04038	0.001293	0.02482	0.01985	0.00497	0.5	0.020	100
0.02104	0.005930	0.05059	0.04808	0.00251	0.5	0.010	150
0.03310	0.003822	0.03427	0.03051	0.00376	0.5	0.015	150
0.03755	0.003822	0.02809	0.02305	0.00505	0.5	0.020	150
0.02974	0.002876	0.05125	0.04874	0.00251	0.5	0.010	200
0.03819	0.002876	0.02944	0.02565	0.00379	0.5	0.015	200
0.03879	0.000735	0.02298	0.01792	0.00506	0.5	0.020	200

APPENDIX F. UPDATE OF THE INSPECTION INTERVAL

Within a given level of confidence α, the relationship that governs the upper prediction limit of the percentage of the population in a failed state at a given point in time (after the new inspection interval is established), F_U, depends on the population size, N, the percentage of the population found failed at inspection, R, and the previous and new inspection intervals, T_1 and T_2. In addition to determining the length of the inspection intervals, adjustment of the new interval is required periodically to maintain or to change a given upper prediction limit. When the procedure is instituted, the upper prediction limit and the initial inspection interval may be based on past practice or some other reasonable engineering estimate. Data collected during the initial inspection will determine the length of the new inspection interval, which will remain in effect until replaced by a regular or emergency update. Procedures to monitor the need to update are in effect at all times and are outlined below.

The following is a possible suggested approach offered without further analysis to choose or update the length of the inspection interval or the equivalent inspection frequency.

F.1 Regular Update

A regular update is performed when two inspection intervals have been completed since (1) the original calculation of the inspection interval, (2) the last regular update, or (3) the last change in length of inspection interval resulting from an emergency update.

At a regular update, the length of the inspection interval in months (T_2) is recalculated in months (using the formula for F_U in Appendix C.1, Step 11) based on the data accumulated during the most recent inspection interval, i.e., the most recent T_2 months.

F.2 Emergency Update

At each monthly inspection, the percentage of units found failed during the four most recent monthly inspections should be reviewed. An emergency update should be performed when the percentage found failed for those four inspections R_4 is greater than

$$R_4 > R_o + Z_\gamma \sqrt{\frac{R_0}{N_4}} \quad .$$

where,

R_4 is the percentage found failed in the last 4 months,

$N_4 = 4N/T_2$ is the number of units inspected during the 4-month interval,

R_o is the percentage R estimated at the last update and,

Z_γ is a standard normal deviate to be chosen (see Notes 1 and 2 below) so that emergency updates caused by random variations are rare,

The emergency update should consist of three steps:

1. Recalculate the length of the required inspection interval based on data from the inspections performed during the latest $T_2/2$ months.

2. After 6 additional months, recalculate the length of the inspection interval based on the latest 12 months of data.

3. Resume monitoring the criterion for an emergency update and await the next regular update.

Note 1: The quantity, γ, to use in determining the standard normal deviate, Z_γ, is still to be determined (possibly with the use of simulations). Some "back of the envelope" analysis suggests that $\gamma \approx 1/(4n)$ may be a good value to try in the simulations.

Note 2: An emergency update should not be performed when the procedure calls for lengthening the inspection interval. Further, all increases in inspection interval at regular updates should be for only half the indicated amount.

Note 3. The formula for calculating the new inspection interval may be used in circumstances where, unlike those circumstances in Appendix C, different amounts of data, based on other than one observation for each unit in the population, is used to determine the percentage of units found failed at inspection. It can be based on QN observations where Q may be greater or less than a significant fraction of 1. If, during the inspection interval T_1, the data collected over a fraction of the time interval of QT_1 is to be used, the total number of observations for failure is QN. In this case, N is replaced by QN in the calculation of S_R. (Therefore, all calculations except Step 10 in Appendix C.1 remain unchanged.) The derivation of the formula for the case when Q is not 1 is changed only by the substitution of QN for N. Note that the simulations of Section 5 and the appendices all refer to the case Q=1. There is no reason to assume that they would change much.

APPENDIX G. COMMENTS

This paper is based on the assumption that it is desired to ensure the actual percentage of units failed in service to be less than some amount (e.g., 5%) with some level of confidence (e.g., 98%) for each population at a given point in time in the future with the prescribed inspection schedule in effect. This requirement is less demanding than requiring the expected value of units failed in service to be less than 5% with 98% confidence for each population. Even less demanding is the requirement that the expected value of the units failed in service over the sum of a number of populations is to be less than 5% with 98% confidence and can be achieved in such a way that each population contributes in achieving the goal. The most demanding requirement is studied in this paper and achieves the safety which has a guarantee not just in general but under the particular circumstances.

Finally, recall that the formulas are derived for a specific idealized model. A most important assumption from a practical point of view is the constant failure rate. This paper does not deal with the consequences of failure rates that vary with time and whether or not repair restores failure rates to their initial values. Even to demarcate those cases where the predictions are conservative from those where they are not would be difficult. All that can be done here is to warn that predictions using these formulas may be inaccurate to an extent determined by the variability of the failure rates.